# General Relativity

## Day 30

April 15th, 2016

**Today**

I Last time

II What is the metric as a map?

III Raising and lowering indices in general and tensor transformation laws.

I. • A tensor is a multilinear map from a set of vectors and dual vectors to the real numbers.

• Why are tensors ubiquitous?

(i) Linearity is an effective initial physical model.

(ii) Linearity is computationally tractable.

(iii) Tensors capture a large class of invariants

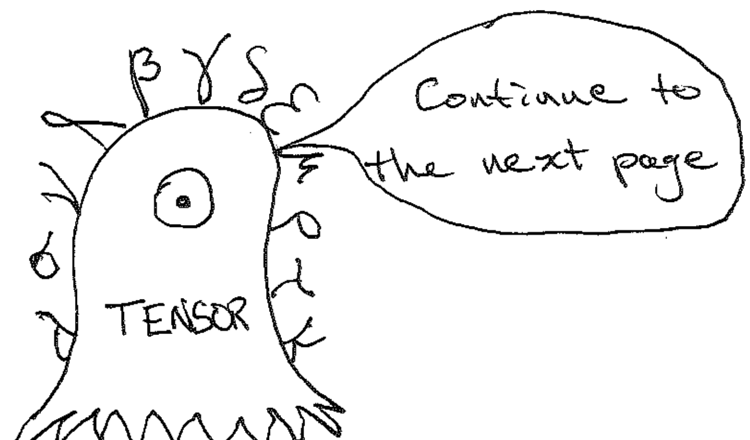
• Vector space $V$

dual space $V^*$

II Consider a linear map

$$M: V \to V.$$

We often associate these maps with matrices. So, is the metric a matrix? It turns out this question needs refinement. We'll build up a little more notation to do this.




... Baffling Beasts Bristling with indices...

After we choose a basis, say $\{\underset{\sim}{e}_\alpha\}$, for $V$ then we have, $M: V \to V$ and

$$a^\alpha \mapsto M^\beta{}_\alpha a^\alpha$$

The array $M^\beta_\alpha$ is what we typically call the matrix representation of the map $M$.

Now, we can revise your question:

Q: Is the metric a linear map from $V$ to $V$? A: No! <u>But</u>, in addition

displaying the numbers $g_{\alpha\beta}$ in a two-dimensional array

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & \text{etc.} & \\ \vdots & & \ddots & \end{pmatrix}$$

There are four types of rank 2 tensors:

(1) $M: V \to V$ or $M^{\beta}{}_{\cdot\alpha}$ ← 1st index ← 2nd index

(2) $g: V \to V^*$ or $g_{\alpha\beta}$

to

$$g(\underset{\sim}{a}, \underset{\sim}{b}) = g_{\alpha\beta} a^\alpha b^\beta$$

we also have

$$a_\alpha = g_{\alpha\beta} a^\beta.$$

So it is a map

$$g: V \to V^*$$

and as long as we remember what space the image is in there is no problem with

(3) $g^{-1}: V^* \to V$ or $g^{\alpha\beta}$

(4) $M^*: V^* \to V^*$ or $(M^*)_\beta{}^\alpha$

III We can raise and lower indices on a general tensor too — use the metric. Suppose $\underset{\sim}{S}$ eats a vector and two dual vectors, then

dual vec.s

$$S(\underset{\sim}{a}, \underset{\sim}{\omega}, \underset{\sim}{\lambda}) = S_\alpha{}^{\beta\gamma} a^\alpha \omega_\beta \lambda_\gamma$$

Then,

$$S(\underset{\sim}{a}, \underset{\sim}{\omega}, \underset{\sim}{\lambda}) = S_\alpha{}^{\beta\gamma}\, a^\alpha \omega_\beta \lambda_\gamma$$

$$= S_{\alpha\beta\gamma}\, a^\alpha \omega^\beta \lambda^\gamma = S_{\alpha\mu\nu}\, a^\alpha g^{\mu\beta}\omega_\beta\, g^{\nu\gamma}\lambda_\gamma$$

So,

$$S_\alpha{}^{\beta\gamma} = S_{\alpha\mu\nu}\, g^{\mu\beta} g^{\nu\gamma}$$

How do you construct a tensor? There are many ways but one nice example is to do it out of vectors:

$$r^{\alpha\beta\gamma} = u^\alpha v^\beta w^\gamma$$

is a rank 3 tensor

Q: What is the essential reason this is a tensor?

Because of its manifest multilinearity (recall that vectors eat dual vectors in a linear manner, $u(\underset{\sim}{\lambda}) = u^\alpha \lambda_\alpha$.)

This is called the "outer" or "tensor" product of vectors. (viz. Quantum Mech.)

Transformation: Suppose we make a coord. transformation from $x^\alpha$ coord.s to $x'^\beta$ coord.s, what happens to a tensor? Well, take eg. the metric $g$,

$$g'_{\alpha\beta} = g(\underset{\sim}{e}'_\alpha, \underset{\sim}{e}'_\beta)$$

$$= g\left(\frac{\partial}{\partial x'^\alpha}, \frac{\partial}{\partial x'^\beta}\right)$$

$$= g\left(\frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial}{\partial x^\gamma}, \frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial}{\partial x^\delta}\right)$$

$$= \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}\, g\left(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\delta}\right)$$

$$= \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}\, g_{\gamma\delta}$$

Same as before!

In general each index gets an appropriate factor of $\frac{\partial x}{\partial x'}$, or $\frac{\partial x'}{\partial x}$.