General Relativity

April 20, 2016

Day 32

Today:

I Last time

II Using Covariant Derivatives

III Curvature Begins

0. Announcements

$$\nabla_{\underset{\sim}{t}} \underset{\sim}{V} (x^\gamma) = \lim_{\epsilon \to 0} \frac{[\underset{\sim}{V}(x^\gamma + \epsilon t^\gamma)]_{\parallel \text{ trans to } x^\gamma} - \underset{\sim}{V}(x^\gamma)}{\epsilon}$$

Using the key recognition that the change upon parallel transport should be proportional to v^γ and to ϵt^γ we found

$$\boxed{\nabla_\beta v^\gamma = \frac{\partial v^\gamma}{\partial x^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma} v^\gamma}$$

means $\underset{\sim}{t} = \underset{\sim}{e}_\beta$

I Last time

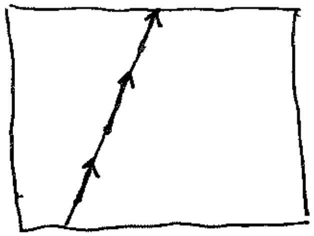
- Tensors are multilinear maps from a collection of vectors and dual vectors to a real number.
- Raise & lower tensor indices with the metric
- Defined the covariant derivative:

II Using Covariant Derivatives

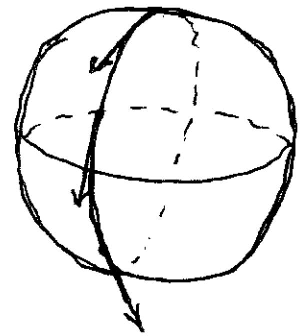
To use the previous formula we need to identify $\tilde{\Gamma}^\alpha_{\beta\gamma}$.

In addition to being extremal, geodesics are as straight as possible — meaning their tangent vectors parallel propogate into each other:

Pictorially,



or



This means that

$$(\nabla_{\underset{\sim}{u}} \underset{\sim}{u})^\alpha = 0 = u^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma} u^\gamma \right)$$

and comparing this with the

In special relativity we defined

$$a^\alpha = \frac{du^\alpha}{d\tau} \quad \text{(LIF only)}$$

but more generally we should define acceleration as something that takes you off of geodesics, i.e.,

$$\underset{\sim}{a} = \nabla_{\underset{\sim}{u}} \underset{\sim}{u}$$

This is sensible because it tells you how the 4-velocity changes

geodesic equation we find,

$$u^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} u^\gamma \right)$$

that $\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}$! This follows because at a point P there are geodesics in every possible direction u^γ and we can identify coefficients.

Let's do some calculations:

as you move in the direction of the 4-velocity! Also in an LIF we find

$$(\nabla_{\underset{\sim}{u}} \underset{\sim}{u})^\alpha = u^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + \overset{\text{0 b/c LIF}}{\tilde{\Gamma}^\alpha_{\beta\gamma}} u^\gamma \right)$$

$$= \frac{dx^\beta}{d\tau} \frac{\partial u^\alpha}{\partial x^\beta} = \frac{du^\alpha}{d\tau} = \overset{\text{previous def.}}{a^\alpha} \checkmark$$

What acceleration is necessary to be stationary in Schwarzschild?

Well, stationary observer has,

$$(\underset{\sim}{u})^\alpha = (u^t, 0, 0, 0)$$

$$\Rightarrow -\left(1-\frac{2M}{r}\right)(u^t)^2 = -1 \Rightarrow u^t = \left(1-\frac{2M}{r}\right)^{-1/2}$$

and,

$$a^\alpha = u^\beta \nabla_\beta u^\alpha = u^t \nabla_t u^\alpha = u^t\left(\frac{\partial u^\alpha}{\partial t} + \Gamma^\alpha_{t\gamma} u^\gamma\right)$$

$$= u^t\left(\cancelto{0}{\frac{\partial u^\alpha}{\partial t}} + \Gamma^\alpha_{tt} u^t\right)$$

Using the Christoffel symbols we have

$$a^\alpha = (0, \Gamma^r_{tt} u^{t2}, 0, 0) = (0, \frac{M}{r^2}, 0, 0)$$

the other objects we've been thinking abt:

$$\nabla_\alpha f \equiv \frac{\partial f}{\partial x^\alpha}, \quad \nabla_{\underset{\sim}{u}} f \equiv u^\alpha \frac{\partial f}{\partial x^\alpha}$$

Now use the product rule (fancier to call it the Leibniz rule):

$$\nabla_\gamma v^\alpha w^\beta = v^\alpha \nabla_\gamma w^\beta + \nabla_\gamma(v^\alpha) w^\beta$$

But $v^\alpha w^\beta$ is a rank two tensor, so,

$$\nabla_\gamma t^{\alpha\beta} = \frac{\partial t^{\alpha\beta}}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^{\delta\beta} + \Gamma^\beta_{\gamma\delta} t^{\alpha\delta}$$

Interesting, it seems well behaved at $r=2M$, what's up?

Well,

$$(\underset{\sim}{a}\cdot\underset{\sim}{a})^{1/2} = \left(1-\frac{2M}{r}\right)^{-1/2} \frac{M}{r^2}$$

still diverges there.

Covariant Derivs of everything else:

Once you have a sensible definition for vectors, you can extend to

just like for $v^\alpha w^\beta$.

$$\nabla_\alpha v_\beta = \frac{\partial v_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma$$

and

$$\nabla_\gamma t^\alpha{}_\beta = \frac{\partial t^\alpha{}_\beta}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^\delta{}_\beta - \Gamma^\delta_{\gamma\beta} t^\alpha{}_\delta \quad \text{etc.}$$