General Relativity

April 22nd, 2016

Day 33

Today's Outline

I Last time

II Curvature in general

I Last time

- Interpretted the geodesic equation as

$$\boxed{\nabla_{\underset{\sim}{u}} \underset{\sim}{u} = 0} \left(= \underset{\sim}{a}\right)$$

which says the tangent to a geodesic is parallel transported into itself along the geodesic.

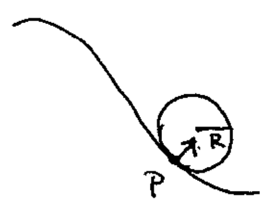
- Extended the covariant derivative to a general tensor,

$$\nabla_\gamma t^\alpha{}_\beta = \frac{\partial t^\alpha{}_\beta}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^\delta{}_\beta - \Gamma^\delta_{\gamma\beta} t^\alpha{}_\delta$$

- Introduced curvature of curves,

$$k = \frac{1}{R}\hat{R}$$

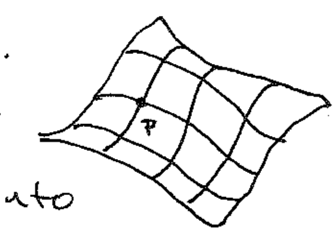
$\uparrow$ (points toward center)



II Curvature in general

Curvature of a Surface

Strategy: draw a line (in the Surface) through P.



Its curvature vector can be resolved into a component $\perp$ to the surface (the "normal" curvature) and a component $\parallel$ to the ~~surf.~~ surf. ("geodesic" curvature)

$$\boxed{\vec{k} = \underbrace{K\hat{n}}_{\text{Normal}} + \underbrace{\vec{k}_g}_{\text{geodesic}}}$$

The normal curv. is the same for all curves passing through P in the same direction — $\vec{k}_g$ is different.

Of course, the normal curv. does depend on the direction through p. Let K_1 be the maximum normal curv., and K_2 be the minimum.

Let α be the angle away from the max direction.

Euler's Formula:

$$\boxed{K(\alpha) = K_1 \cos^2\alpha + K_2 \sin^2\alpha}$$

(the max & min are at 90°)

K_1 and K_2 called the principal normal curvatures.

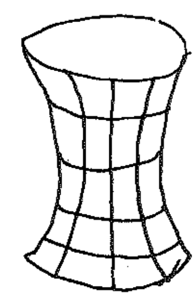
Ambiguity: $\hat{n}$ could be "up" or "down"

Convention: draw $\hat{n}$ such that K_1 (the max normal curv.) is positive, then K_2 could be pos., neg. or zero.

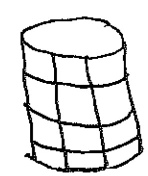
Classification:

(1) If K_2 is pos., P is called an elliptic pt. (e.g. any point on an ellipsoid)

(2) If K_2 is negative P is called a hyperbolic pt



(3) If one (or both) is zero P is a parabolic pt. (e.g. any pt in a plane or any pt on a cylinder)



If $K_1 = K_2$ the pt. p is called a <u>navel</u>.

<u>Examples</u>: (1) cylinder: $K_1 = \frac{1}{R}$, $K_2 = 0$.

(2) Sphere : $K_1 = K_2 = \frac{1}{R}$

(3) A general ellipsoid has four navels. (can have more, but in general there are 4).

Extrinsic / Intrinsic Curvature

Looking at a surface from point of view of imbedding in 3 space.

Intrinsic: From point of view of someone constrained to work in surface – can measure distance, angles, but no access to 3rd dim.

Intrinsic geom. unchanged by "rolling" – "bending" (as opposed to stretching or tearing).

Gauss & Bending Invariants

Note that the principal normal curvatures are <u>not</u> bending invariants. But their product, $K_1 K_2$, is a bending invariant, call it K. (Gaussian Curvature) Remarkable theorem whose proof you should check out.

This leads to a remarkable idea – can we characterize curvature completely using bending invariants (i.e. intrinsically).

Riemann's answer : Yes!

In n dimensions there are $\frac{1}{12} n^2 (n^2 - 1)$ curvatures that can be collected into a tensor.

Table

dim n	type	intrinsic curvatures
1	Line	0
2	Surface	$\frac{1}{12} 4 \cdot 3 = 1$
3	Space	$\frac{1}{12} 9 \cdot 8 = 6$
4	Space time	$\frac{1}{12} 16 \cdot 15 = 20$

captured by Riemann curvature tensor

P4/4

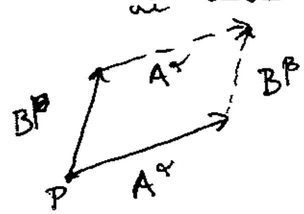
Riemann Tensor

How do we get at these invariants? There are (atleast) two ways:

(1) In flat space geodesics that are initially $\parallel$ remain so, this changes in curved spaces

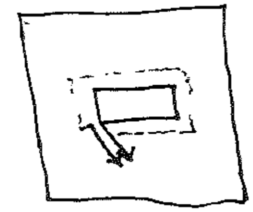
Hartle does (1), we'll do (2).

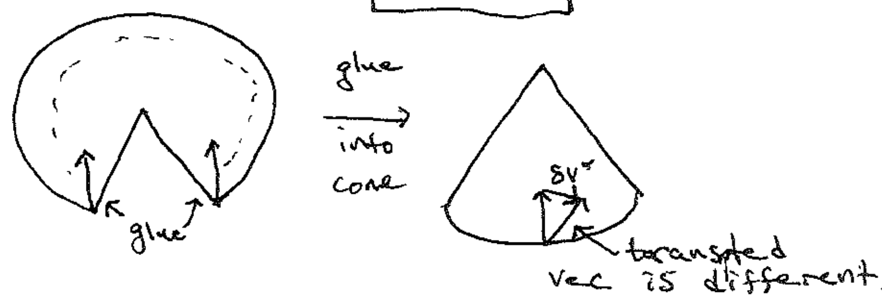
More precisely let's $\parallel$-transport around a parallelogram whose edges are $\underset{\sim}{A}$ and $\underset{\sim}{B}$



If we carry a vector $\underset{\sim}{V}$ around the loop its change upon returning to the initial pt is

(2) In flat space if you $\parallel$-transport a vector around a loop it is unaffected. Not true in a curved space





$$V^\alpha_{\substack{\parallel\text{-trans.}\\ \text{around loop}}}(P) - V^\alpha(P) = \delta V^\alpha(P)$$

We want to characterize the curvature at the pt. P and so the vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ should be small, so let $A^\alpha = dx^\alpha$ and $B^\beta = dy^\beta$ then the Riemann tensor is a tensor that takes V^β, dx^γ, dy^δ as input and returns δV^α:

$$\boxed{\delta V^\alpha = - R^\alpha{}_{\beta\gamma\delta} V^\beta dx^\gamma dy^\delta}$$

To Be continued...