General Relativity

Day 36

Today's Outline

I Last time

II Connecting to the tensor formulation of the Einstein equation

To capture full Einstein eqn we require this to hold for balls of test particles initially at rest in all inertial frames.

- We began the process of connecting this formulation with the full tensorial one:

Step 1: showed $\lim_{\epsilon \to 0} \frac{a^\alpha}{\epsilon} = - R^\alpha{}_{\beta\gamma\delta} v^\beta u^\gamma v^\delta$ the geodesic deviation equation.

I Last lecture

- Introduced the Einstein equation in the form:

$$\frac{\ddot{V}}{V}\bigg|_{t=0} = -4\pi \begin{pmatrix} \text{flow of } t\text{-mom. in } t \text{ dir.} \\ + \text{ " } \quad x\text{-mom in } x \text{ dir.} \\ + \text{ " } \quad y\text{-mom in } y \text{ dir.} \\ + \text{ " } \quad z\text{-mom in } z \text{ dir.} \end{pmatrix}$$

$$= -4\pi \left(\rho + P_x + P_y + P_z \right)$$

(units w/ $G = c = 1$).

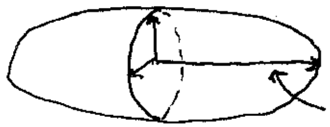
II Tensor Formulation

We continue with Step 2: Acceleration to time evolution of the volume.

Choose a LIF centered on the ball and such that the center doesn't accelerate to 2nd order in time. We know,

ball $\rightarrow$ ellipsoid,

so choose axes of the frame such that they align with the axes of the ellipsoid and let $r^j(t)$ $(j=1,2,3)$ be the radii of these axes,



$r^2(t)$

If the initial ball radius is ϵ, we have

$$r^j(t) = \epsilon + \tfrac{1}{2} a^j t^2 + O(t^3)$$

and so,

$$\lim_{t\to 0} \frac{\ddot{r}^j}{r^j} = \lim_{t\to 0} \frac{a^j}{\epsilon}$$

Because $V \propto r_1 r_2 r_3$ we have

$$\frac{\ddot{V}}{V} \xrightarrow[t\to 0]{} \frac{\ddot{r}^1}{r^1} + \frac{\ddot{r}^2}{r^2} + \frac{\ddot{r}^3}{r^3}$$

(the cross terms involving $\dot{r}^j$ fall out as $t\to 0$ because we assumed $\dot{r}^j(0)=0$).

So,

$$\lim_{\underbrace{V\to 0}_{\text{condition for LIF}}} \left.\frac{\ddot{V}}{V}\right|_{t=0} = -R^\alpha{}_{t\alpha t} \quad \begin{pmatrix}\text{yes, sum}\\ \text{on } \alpha\end{pmatrix}$$

The sum is over all values of α $(\alpha = 0,1,2,3)$

With this setup the initial 4-velocity of the ball is purely timelike $U^\alpha = (1,0,0,0)$ and and we can take $\underset{\sim}{u}$ to be each coord. axis one at a time, so that,

$$\lim_{t\to 0} \frac{\ddot{r}^j(t)}{r^j(t)} = -R^j{}_{\beta j \delta} U^\beta U^\delta \quad \begin{pmatrix}\text{no}\\ \text{sum}\\ \text{on } j\end{pmatrix}$$

$$= -R^j{}_{tjt} \quad \begin{pmatrix}\text{no sum}\\ \text{on } j\end{pmatrix}$$

because $R^t{}_{ttt} = 0$ and it's easier to write as a sum over all values.

When we contract over two indices of the Riemann tensor in this pattern we call the result the Ricci tensor:

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$$

Ricci tensor

$$= R^0{}_{\alpha 0\beta} + R^1{}_{\alpha 1\beta} + R^2{}_{\alpha 2\beta} + R^3{}_{\alpha 3\beta}.$$

So, we've found, time-time component of the Ricci tensor

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -R_{tt}$$

Our ball of test particles is responding to the curvature of spacetime!

Step 3: Matter-Energy-Pressure Causes Curvature

A facet of Einstein's insight was that Mass-Energy-Pressure causes this curvature. Simply stated, the ball accelerates due to gravitational attraction, that is,

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -4\pi(\rho + P_x + P_y + P_z)$$

$$= -4\pi(T_{tt} + T_{xx} + T_{yy} + T_{zz})$$

Here I've reintroduced the stress-energy tensor $T_{\alpha\beta}$. As a tensor it eats two vectors - first one tells you which momentum and second in which direction.

Step 4: General Frame

Putting it all together we find

$$R_{tt} = 4\pi(T_{tt} + T_{xx} + T_{yy} + T_{zz})$$

We've been working in an LIF, which has metric $g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$.

It would be convenient to write $T^\gamma{}_\gamma$ but

$$T^\gamma{}_\gamma = -T_{tt} + T_{xx} + T_{yy} + T_{zz}$$

because of metric

But we can fix this with $g_{tt} = -1$,

$$T_{tt} - \tfrac{1}{2} g_{tt} T^\gamma{}_\gamma =$$

$$T_{tt} + \tfrac{1}{2}(-T_{tt} + \cdots + T_{zz})$$

$$= \tfrac{1}{2}(T_{tt} + \cdots + T_{zz})$$

Then we have

$$R_{tt} = 8\pi\left(T_{tt} - \tfrac{1}{2} g_{tt} T^\gamma{}_\gamma\right)$$

To promote this to any frame we require that it hold as a tensor equation:

$$R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T^\gamma{}_\gamma\right)$$

Our first version of Einstein's eqn!
Traditionally people prefer the matter side be simple, so note,

$$R^{\beta}{}_{\beta} = 8\pi\left(T^{\beta}{}_{\beta} - \tfrac{1}{2} g^{\beta}{}_{\beta} T^{\gamma}{}_{\gamma}\right)$$

$$= 8\pi\left(T^{\beta}{}_{\beta} - \tfrac{1}{2}\cdot 4 \cdot T^{\beta}{}_{\beta}\right)$$

$$= -8\pi T^{\beta}{}_{\beta}$$

We call $R^{\beta}{}_{\beta} \equiv R$ the Ricci scalar. Putting this back

$$\boxed{G_{\alpha\beta} = 8\pi T_{\alpha\beta}}$$

In units with G and c,

$$\boxed{G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}$$

A remarkable, perhaps the most remarkable, aspect of these eqns is that they are nonlinear.

into the E eqn we find,

$$R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}\left(-\tfrac{1}{8\pi}R\right)\right)$$

$$\Rightarrow R_{\alpha\beta} = 8\pi T_{\alpha\beta} + \tfrac{1}{2} g_{\alpha\beta} R$$

$$\Rightarrow \boxed{R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}}$$

The Einstein Equation!

Abbreviations:

Einstein tensor $\Rightarrow G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R$

$$R^{\alpha}{}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\gamma\epsilon}\Gamma^{\epsilon}_{\beta\delta} - \Gamma^{\alpha}_{\delta\epsilon}\Gamma^{\epsilon}_{\beta\gamma}$$

$$R^{\gamma}{}_{\alpha\gamma\beta} = R_{\alpha\beta} = \frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\gamma\epsilon}\Gamma^{\epsilon}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\epsilon}\Gamma^{\epsilon}_{\alpha\gamma} \quad \left(\text{sum on } \gamma\right)$$

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right)$$

$G_{\alpha\beta} \ni$ (contains) 2nd deriv.s of g, 1st derivs of g and many nonlinear terms.