General Relativity

Day 37

Today

I Last time

II Solve the spherically symmetric, static vacuum Einstein equation

III Begin gravitational waves

$$\boxed{G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta},}$$

where

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R$$

is the Einstein tensor and

$$R^{\alpha}{}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\gamma\epsilon}\Gamma^{\epsilon}_{\beta\delta} - \Gamma^{\alpha}_{\delta\epsilon}\Gamma^{\epsilon}_{\beta\gamma},$$

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta},$$

$$R = R^{\alpha}{}_{\alpha}.$$

I • Starting from a ball of test particles satisfying

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -4\pi\left(\rho + P_x + P_y + P_z\right)$$

we derived the Einstein eqn:

$$\boxed{R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T^{\gamma}{}_{\gamma}\right)}$$

• We recast this in the form

Finally,

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right)$$

So, $G_{\alpha\beta}$ contains 2nd derivatives of g, 1st derivatives of g and many nonlinear terms.

• Using the Moore-Rindler worksheet we found all the components of the Ricci tensor $R_{\alpha\beta}$ for a spherically symmetric, static vacuum spacetime.

II From the Moore-Rindler worksheet:

$$R_{00} = R_{tt} = \frac{1}{2B}\left[\frac{d^2A}{dr^2} - \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 - \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r}\frac{dA}{dr}\right]$$

$$R_{11} = R_{rr} = \frac{1}{2A}\left[-\frac{d^2A}{dr^2} + \frac{1}{2A}\left(\frac{dA}{dr}\right)^2 + \frac{1}{2B}\frac{dA}{dr}\frac{dB}{dr} + \frac{2A}{rB}\frac{dB}{dr}\right]$$

$$R_{22} = R_{\theta\theta} = -\frac{r}{2AB}\frac{dA}{dr} + \frac{r}{2B^2}\frac{dB}{dr} + 1 - \frac{1}{B}$$

$$R_{33} = R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

In vacuum the Einstein equation is $R_{\alpha\beta} = 0$, so we want these all to vanish. Then

$$0 = 2B R_{tt} + 2A R_{rr} = \frac{2}{r}\frac{dA}{dr} + \frac{2A}{rB}\frac{dB}{dr},$$

as long as $A \neq 0$ and $B \neq 0$. If we also take $r \neq 0$ this is

$$B\frac{dA}{dr} + A\frac{dB}{dr} = 0$$

$$\Rightarrow \frac{d}{dr}(AB) = 0$$

and

$$AB = \text{const.}$$

We'd like to fix the value of this constant, but how? The idea is to use a boundary condition; far from any gravitating bodies the metric should become that of flat spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2.$$

This means that asymptotically $AB = 1$ and since it is constant it must always have this value. Then

$$B = \frac{1}{A},$$

$$A\frac{dB}{dr} = -B\frac{dA}{dr} \Rightarrow \frac{dB}{dr} = -\frac{1}{A^2}\frac{dA}{dr},$$

and $R_{22} = 0$ implies

$$-\frac{r}{2}\frac{dA}{dr} - \frac{r}{2}\frac{dA}{dr} + 1 - A = 0$$

or

$$r\frac{dA}{dr} + A = 1$$

$$\Rightarrow \frac{d}{dr}(rA) = 1$$

Integrating yields

$$r A + \underbrace{R_s}_{\text{const. of int.}} = r$$

$$\Rightarrow \boxed{A = 1 - \frac{R_s}{r}}$$

and

$$\boxed{B = \left(1 - \frac{R_s}{r}\right)^{-1}}.$$

So,

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

is a solution! On the HW you will prove

$$R_s = \frac{2GM}{c^2}.$$

III We've just found an exact solution to the EE. Let's start to look for approximate solutions too!

A good idea is to take

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

metric perturbations all assumed to be small compared to 1