General Relativity

Day 38

<u>Today</u>

I The linearized Einstein equation

II Gauge choices and the wave equation

III Solving the wave equation

and since $R_{\alpha\beta}$ is constructed out of derivatives of the metric we know that the constant η will contribute nothing to this eqn. We now compute the contribution of h: First note that if we are to neglect quadratic and higher powers of h, then $\eta^{\alpha\beta}$ is sufficient for raising indices. To lowest order then

change from flat spacetime $\to$ $$\delta\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left(\frac{\partial h_{\delta\alpha}}{\partial x^{\beta}} + \frac{\partial h_{\delta\beta}}{\partial x^{\alpha}} - \frac{\partial h_{\alpha\beta}}{\partial x^{\delta}}\right).$$

I Last time we intro'd the idea of a metric perturbation:

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x)$$

where all components of h are small compared to one.

In vacuum the Einstein eqn is

$$R_{\alpha\beta} = 0$$

The Ricci tensor has terms like $\partial\Gamma/\partial x$ and like $\Gamma\cdot\Gamma$, but the second type are negligible as they go like h^2. So,

$$\delta R_{\alpha\beta} = \frac{\partial\,\delta\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial\,\delta\Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}}$$

while, if we let $\partial/\partial x^{\gamma} \equiv \partial_{\gamma}$,

$$\partial_{\gamma}\delta\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left(\partial_{\gamma}\partial_{\beta}h_{\delta\alpha} + \partial_{\gamma}\partial_{\alpha}h_{\delta\beta} - \partial_{\gamma}\partial_{\delta}h_{\alpha\beta}\right)$$

and

$$\partial_{\beta}\delta\Gamma^{\gamma}_{\alpha\gamma} = \frac{1}{2}\eta^{\gamma\delta}\left(\partial_{\beta}\partial_{\gamma}h_{\delta\alpha} + \partial_{\beta}\partial_{\alpha}h_{\delta\gamma} - \partial_{\beta}\partial_{\delta}h_{\alpha\gamma}\right)$$

Then

$$\delta R_{\alpha\beta} = \frac{1}{2}\Big[-\partial_\gamma\partial^\gamma h_{\alpha\beta} + \partial_\alpha\partial_\gamma h^\gamma{}_\beta + \partial_\beta\partial^\gamma h_{\gamma\alpha} - \partial_\alpha\partial_\beta h^\gamma{}_\gamma\Big]$$

Now, let

$$\Box = \partial_\gamma\partial^\gamma = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2$$

and

$$V_\alpha \equiv \partial_\gamma h^\gamma{}_\alpha - \frac{1}{2}\partial_\alpha h^\gamma{}_\gamma$$

Then

$$\delta R_{\alpha\beta} = \frac{1}{2}\Big[-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha\Big]$$

Can we make it so?

II As always in G.R. we are free to change our coordinate system. We would like to make a change that preserves $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, but changes the h's. Consider

$$x'^\alpha = x^\alpha + \xi^\alpha(x),$$

where the ξ are required to be small, like the h's, but otherwise arbitrary.

[check:

$$\partial_\alpha V_\beta = \partial_\alpha\partial_\gamma h^\gamma{}_\beta - \frac{1}{2}\partial_\alpha\partial_\beta h^\gamma{}_\gamma$$

$$\partial_\beta V_\alpha = \partial_\beta\partial_\gamma h^\gamma{}_\alpha - \frac{1}{2}\partial_\alpha\partial_\beta h^\gamma{}_\gamma$$

$$= \partial_\beta\partial^\gamma h_{\gamma\alpha} - \frac{1}{2}\partial_\alpha\partial_\beta h^\gamma{}_\gamma$$

It works.]

The linearized Einstein equation is then

$$\delta R_{\alpha\beta} = 0,$$

which is almost a wave equation for $h_{\alpha\beta}$, namely, $\Box h_{\alpha\beta} = 0$.

Note that

$$x^\alpha = x'^\alpha - \xi^\alpha(x^\beta)$$

$$\simeq x'^\alpha - \Big[\xi^\alpha(x'^\beta) + \underbrace{\frac{d\xi^\alpha}{dx^\beta}\Big|_{x'^\beta}(x^\beta - x'^\beta) + \dots}_{2^{nd}\text{ order in }\xi}\Big]$$

$$\simeq x'^\alpha - \xi^\alpha(x'^\beta)$$

In small quantities you can freely switch between coordinate systems.

Then, $\dfrac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha_\beta - \dfrac{\partial \xi^\alpha}{\partial x'^\beta} \simeq \delta^\alpha_\beta - \dfrac{\partial \xi^\alpha}{\partial x^\beta}$.

Then we also have,

$$g'_{\alpha\beta}(x') = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta}(x)$$

$$= \left(\delta^\gamma_\alpha - \frac{\partial \xi^\gamma}{\partial x^\alpha}\right)\left(\delta^\delta_\beta - \frac{\partial \xi^\delta}{\partial x^\beta}\right)\left(\eta_{\gamma\delta} + h_{\gamma\delta}\right)$$

$$= \eta_{\alpha\beta} + h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta$$

Then

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta$$

The functions $\xi_\alpha(x)$ are arbitrary and four in number, so we can choose them exactly to kill off the four V_α, so that

$$V'_\alpha(x) = 0$$

Let's assume we are in such a coord. system, then,

$$\Box h_{\alpha\beta}(x) = 0$$

and

$$V_\alpha(x) = \partial_\beta h^\beta_\alpha(x) - \tfrac{1}{2}\partial_\alpha h^\beta_\beta(x) = 0,$$

are the two equations to study. The 2nd equation is called a gauge condition — in this case the Lorentz gauge condition.

III The wave equation is

$$\Box f(x) \equiv \eta^{\alpha\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = -\frac{\partial^2 f}{\partial t^2} + \vec{\nabla}^2 f$$

$$= -\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^3} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Any function of the form

$$f = f(t \pm z)$$

is a solution to this equation and represents a wave moving in the z-direction. A class of especially useful solutions is

$$f(x) = a\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} = a\, e^{i(-k^t t + \vec{k}\cdot\vec{x})},$$

which are plane waves. For these

$$\Box f = -\underset{\sim}{k} \cdot \underset{\sim}{k}\, f$$

and this vanishes iff $\underset{\sim}{k}$ is null. So,

$$(\underset{\sim}{k})^\alpha = (|\vec{k}|, \vec{k}).$$