General Relativity — Day 4

Feb 8th, 2016

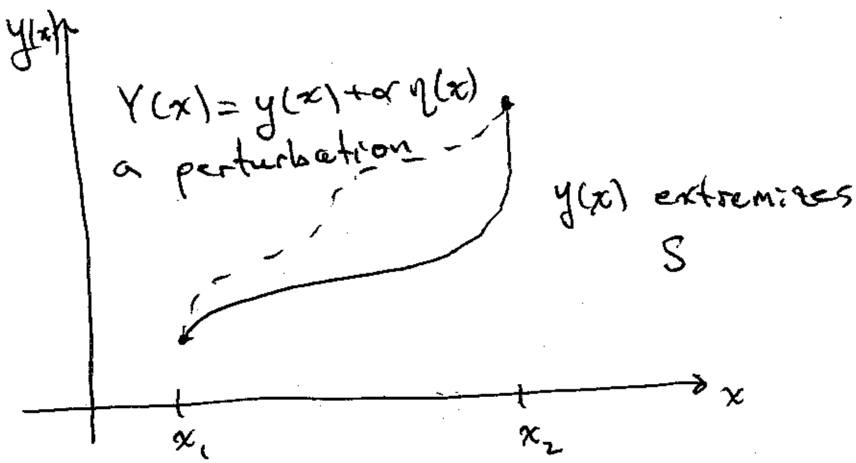
<u>Today</u>

I Last time

II The Euler-Lagrange eqn.s

III The geometry of Special Relativity

I
- Gave our 1st definition of geodesics: curves of extremal length in a given geometry.
- Setup Lagrange's method for finding the extremals of an integral



$$S = \int_{x_1}^{x_2} L(x, y(x), y'(x))\, dx$$

$$Y(x) = y(x) + \alpha\eta(x) \equiv y(x) + \delta y(x)$$

- Key idea: Insert $Y(x)$ into S and look for

$$\boxed{\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0}$$

II We began this calculation last time. If $Y(x) = y(x) + \alpha\eta(x)$ then

$$Y'(x) = y'(x) + \alpha\eta'(x)$$

We have

$$S(\alpha) = \int_{x_1}^{x_2} L(x, Y, Y')\, dx$$

$$= \int_{x_1}^{x_2} L(x, y+\alpha\eta, y'+\alpha\eta')\, dx$$

Then

$$\frac{dS}{d\alpha} = \frac{d}{d\alpha}\int_{x_1}^{x_2} L\, dx = \int_{x_1}^{x_2} \frac{\partial L}{\partial \alpha}\, dx$$

and

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial y'}\eta' \right) dx$$

"You can swith a derivative at the cost of a minus sign and a boundary term"

Then,

$$\int_{x_1}^{x_2} \frac{\partial L}{\partial y'} \cdot \frac{d\eta}{dx}\, dx = \underbrace{\left(\frac{\partial L}{\partial y'}\eta\right)\Big|_{x_1}^{x_2}}_{\substack{=0 \text{ bcse of assumed} \\ \text{bndry cond. on } \eta.}} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\eta\, dx$$

and

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right)\cdot \eta\, dx$$

To simplify the 2nd term we use integration by parts

$$\int_{x_1}^{x_2} \frac{d}{dx}(f\cdot g)\, dx = f(x)\cdot g(x)\Big|_{x_1}^{x_2}$$

(integral = antiderivative). But, also

$$\int_{x_1}^{x_2} \frac{d}{dx}(f\cdot g)\, dx = \int_{x_1}^{x_2} f' \cdot g\, dx + \int_{x_1}^{x_2} f\cdot g'\, dx$$

Putting these together

$$\int_{x_1}^{x_2} \frac{df}{dx} g\, dx = f(x)g(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} f\,\frac{dg}{dx}\, dx.$$

The pinnacle of the α cleverness is that $\frac{dS}{d\alpha} = 0$ is to hold for all η. So, what can we say when

$$\int_{x_1}^{x_2} g(x)\,\eta(x)\, dx = 0$$

for all η? Exercise: Prove that this implies that $g = 0$.

In our case

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0 \quad \text{E-L eqns or Euler-Lagrange eqns}}$$

<u>Ex. 1</u>: Let's return to finding the geodesics of the Euclidean plane. The integral we want to extremize is

$$\text{Length} = S = \int dS = \int \sqrt{dx^2 + dy^2}$$

This is intended as a line integral taken along the curve. This means that we have to choose an independent variable to parametrize the curve. Let's choose $y = y(x)$ again. Then

$$dy = \frac{dy}{dx}\, dx$$

and if we fix the end points x_1 and x_2 we have

$$S = \int_{x_1}^{x_2} \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Hence our Lagrangian for this problem is

$$L = L(x, y(x), y'(x)) = \sqrt{1 + y'^2}.$$

The E-L equations are then

$$\frac{\partial L}{\partial y} = 0 \,; \qquad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

$$\Rightarrow \quad 0 - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

and

$$\frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = \text{const.}$$

But since the l.h.s. is only a function of y' it must be that

$$y' = \text{const.} \underset{\text{call it}}{\equiv} m$$

and integration gives

$$y(x) = mx + b,$$

the equation of a line!

We could impose boundary conditions to find m and b in terms of x_1, x_2, y_1 and y_2.

<u>Ex 2</u>: Another important application of the Calculus of Variations and the E-L eqn.s is to mechanics.

Consider position as a function of time $x(t)$ and the integral

$$S = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left(\tfrac{1}{2}mv^2 - V(x)\right) dt,$$

Here $v(t) = \frac{dx}{dt} \equiv \dot{x}$ is the velocity, $V(x)$ is the potential energy and the Lagrangian is

$$L = \tfrac{1}{2}mv^2 - V.$$

This integral is called the action in physics. The E-L equations are built up out of

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} \equiv F(x),$$

where we recall that the negative of the gradient of the potential is the force, and

$$\frac{\partial L}{\partial v} = \frac{\partial L}{\partial \dot{x}} = mv$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = ma$$

acceleration

Then,

$$ma = F$$

and we have recovered Newton's 2nd law as the E-L equations for the action integral.

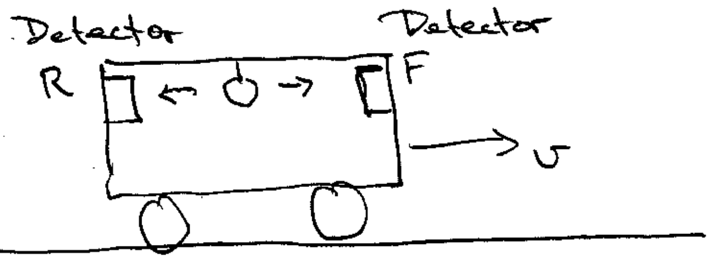
Hamilton realized that this provides an independent axiomatization of mechanics!

A particle moves btwn a pt in space at one time and another pt in space at a later time so as to extremize the action in between. (Hamilton's Principle)

III The study of geometry in special relativity is quite different than in the Euclidean plane. This is due to the presence of time in SR.

Let's start to investigate this difference. According to Einstein the speed of light is the same in all reference frames.

Def.: An event happens at a particular location at a particular time.



Which detector (R or F) fires first?

(A) Observer on the train:

Events R & F are simultaneous.

(B) Observer on the ground

R before F.

Conclusion: Two events simultaneous to one (inertial) observer, may not be to another!

"Observation": What you get after correcting for how long the message took to reach you. (You could think of a custodian as being attached to each reference frame.)