# General Relativity — Day 6

Feb $12^{th}$, 2016

## Today

I Last time

II What are Lorentz transformations?

Spacetime intervals:

$(\Delta s)^2 > 0$ Spacelike separated events

$(\Delta s)^2 = 0$ Lightlike or null "

$(\Delta s)^2 < 0$ Timelike "

- SR Effects

(i) $L = L_*/\gamma$ ; $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

(ii) $dt = \gamma \, d\tau$

(iii) $v_{AC} = \dfrac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}/c^2}$

I • We agreed to:

$\boxed{\text{seek a distance all observers agree upon; an } \underline{\text{invariant}} \text{ distance}}$

and arrived at

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

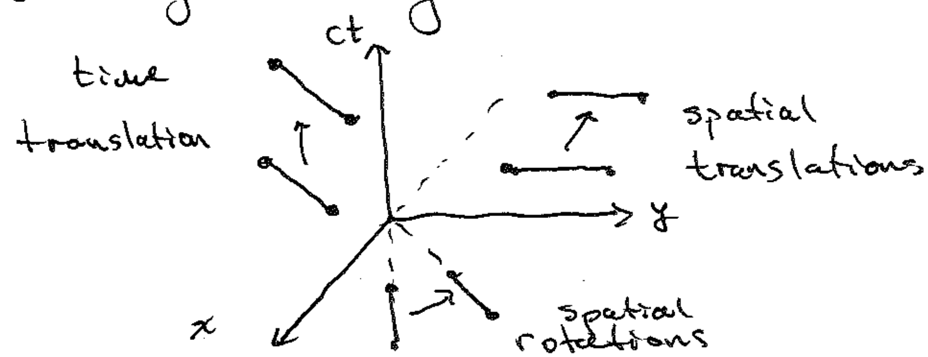
the "spacetime" or "invariant" interval.

- characterized various

II What are Lorentz transformations?

They are the set of all transformations that leave the spacetime interval $ds^2$ invariant.

Some geometrically clear examples



Let's also demonstrate some of these analytically, say rotation:

$$\begin{pmatrix} c\,dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix},$$

where $c\theta \equiv \cos\theta$ and $s\theta \equiv \sin\theta$. Then

$$c\,dt' = c\,dt, \quad dx' = c\theta\,dx - s\theta\,dy$$

$$dy' = s\theta\,dx + c\theta\,dy, \quad dz' = dz$$

Then,

$$ds'^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2$$

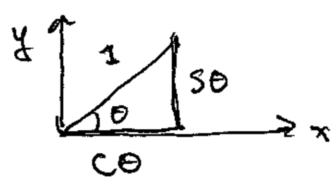
$$= -c^2dt^2 + (c^2\theta\,dx^2 - 2s\theta c\theta\,dx\,dy + s^2\theta\,dy^2) + (s^2\theta\,dx^2 + 2s\theta c\theta\,dx\,dy + c^2\theta\,dy^2) + dz^2$$

$$= -c^2dt^2 + dx^2 + dy^2 + dz^2. \checkmark$$

Can we characterize a transformation that mixes $x$ and $t$ but leaves $-c^2dt^2 + dx^2$ invariant?

For rotations we needed

$$\cos^2\theta + \sin^2\theta = 1$$




We also have hyperbolic versions

$$\cosh^2\theta - \sinh^2\theta = 1$$

where

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}; \quad \sinh = \frac{e^{\theta} - e^{-\theta}}{2}.$$

So, we can consider

$$\begin{pmatrix} c\,dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} ch\theta & -sh\theta & 0 & 0 \\ -sh\theta & ch\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}$$

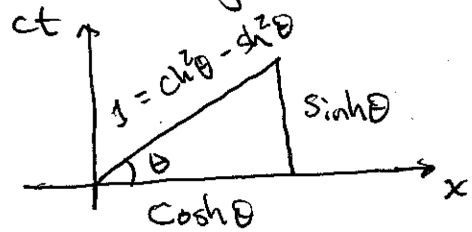
with $ch\theta \equiv \cosh\theta$ and $sh\theta \equiv \sinh\theta$. Now,

$$ds'^2 = -(ch\theta\,c\,dt - sh\theta\,dx)^2 + (-sh\theta\,c\,dt + ch\theta\,dx)^2 + dy^2 + dz^2$$

$$= -(ch^2\theta - sh^2\theta)c^2dt^2 + (ch^2\theta - sh^2\theta)dx^2 + dy^2 + dz^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = ds^2.$$

These additional transformations are called <u>boosts</u>, or <u>Lorentz boosts</u>.

We start to see what our new geometry is! It's The geometry of hyperbolic triangles




Recall,

$$ch^2\theta - sh^2\theta = 1$$

$$\Rightarrow \quad 1 - \tanh^2\theta = \frac{1}{ch^2\theta}$$

So,

$$\beta = \frac{v}{c} = \tanh\theta$$

and

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \tanh^2\theta}}$$

In fact, we can translate the expressions for the boost in terms of $ch\theta$ and $sh\theta$ into expressions in terms of $v$, the boost speed and $\beta \equiv v/c$ and $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$.

The relationship is

$$\boxed{v = c \tanh\theta}$$

But then,

$$\gamma = \frac{1}{\sqrt{\frac{1}{ch^2\theta}}} = \sqrt{ch^2\theta} = ch\theta$$

and

$$\beta \cdot \gamma = \tanh\theta \cdot ch\theta = sh\theta$$

Putting these transformations into the boost we find

$$\begin{pmatrix} c\,dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix},$$

exactly as we claimed before.