General Relativity
Day 7

Today

I Last time

II Relativistic effects from Lorentz transformations

III Units where $c = 1$.

I
- Lorentz transformations are the set of all transf. that preserve the flat spacetime line element $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$.
- In matrix form a boost takes the form

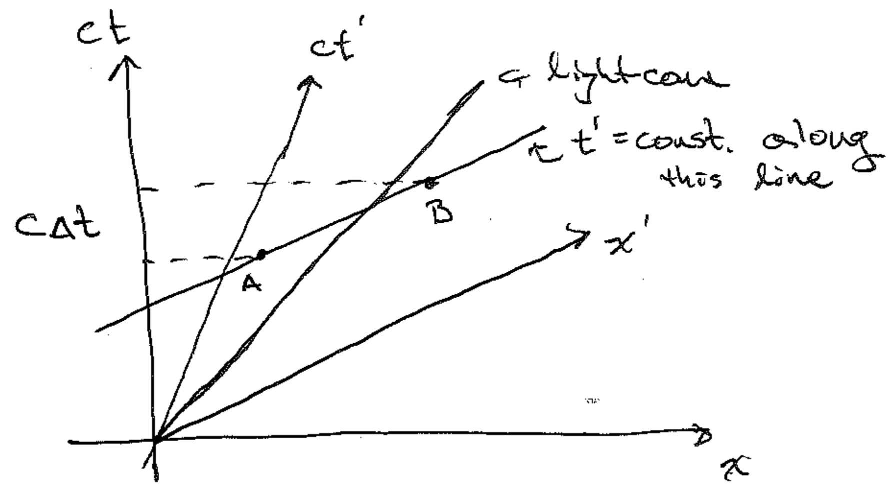
$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} ch\theta & -sh\theta & 0 & 0 \\ -sh\theta & ch\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

or

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$
$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$
$$y' = y$$
$$z' = z$$

where $\frac{v}{c} = \tanh\theta$

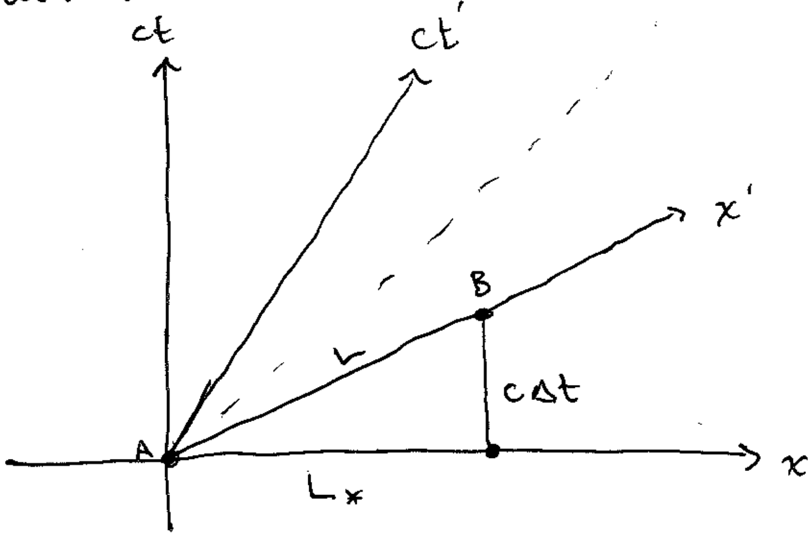
II (i) As we have seen the spacetime diagram depicting a boosted frame is!



Note that $\Delta t' \equiv t'_B - t'_A = 0$, hence using the inverse Lorentz transformation (which we easily obtain by taking $t \leftrightarrow t'$, $x \leftrightarrow x'$ etc and $v \to -v$)

$$c\Delta t = \gamma\left(c\Delta t' + \frac{v}{c}\Delta x'\right)$$
$$= \gamma \frac{v}{c} \Delta x'$$

This is precisely the relativity of simultaneity — two events separated by $\Delta x'$ and such that $\Delta t' = 0$ have non zero time separation, $\Delta t \neq 0$ of time, but the latter depends on frame.



The invariant interval gives

in the unprimed frame.

(ii) Length contraction: Consider a rod of length L_* in its rest frame and call its length L in a frame moving at speed V — this is necessary because the length of a rod is a measurement of the spatial location of its ends at the same moment

us that $\Delta s'^2_{AB} = \Delta s^2_{AB}$ and hence

$$L^2 = -c^2\Delta t^2 + L_*^2$$

But, from the Lorentz trans.

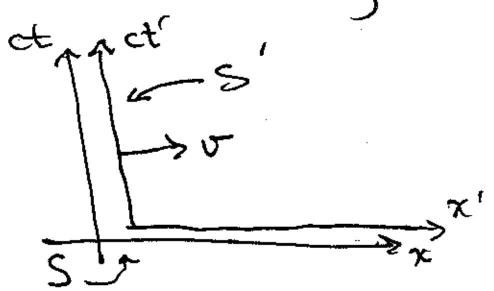
$$t' = 0 \Rightarrow t = \frac{V}{c^2}x \Rightarrow c\Delta t = \frac{V}{c}\Delta x = \frac{V}{c}L_*$$

so that

$$L^2 = L_*^2 - \frac{V^2}{c^2}L_*^2$$
$$= L_*^2\left(1 - \frac{V^2}{c^2}\right)$$
$$= L_*^2 \, 1/\gamma^2$$

or $L = L_*/\gamma$, precisely length contraction

(iii) Velocity addition: Suppose two frames are in relative motion with the second moving at speed v



and that there is a particle moving through space, so that

In S: $x(t), y(t), z(t)$ and

in S': $x'(t'), y'(t'), z'(t')$

provide descriptions of the particle motion. Then

$$V^{x'} \overset{\text{def.}}{\equiv} \frac{dx'}{dt'} \overset{\text{by Lorentz trans.}}{=} \frac{\gamma(dx - \frac{v}{c}c\,dt)}{\frac{1}{c}\gamma(c\,dt - \frac{v}{c}dx)}$$

Dividing numerator and denom. by dt we obtain,

$$V^{x'} = \frac{\gamma(\frac{dx}{dt} - v)}{\gamma(1 - \frac{v}{c^2}\frac{dx}{dt})}$$

$$= \frac{V^x - v}{1 - vV^x/c^2}$$

precisely the Einstein velocity addition rule. In most intros to SR we don't look at $V^{y'}$, so let's do it:

$$V^{y'} \equiv \frac{dy'}{dt'} = \frac{dy}{\frac{1}{c}\gamma(c\,dt - \frac{v}{c}dx)}$$

$$= \frac{dy/dt}{\gamma(1 - \frac{v}{c^2}\frac{dx}{dt})} = \boxed{\frac{V^y}{\gamma\left(1 - \frac{vV^x}{c^2}\right)}}$$

The "transverse" components $V^{y'}$ and $V^{z'}$ transform in a richer way.

III Because all (inertial) observers agree on the speed of light we can use it as a scale for measuring speeds. From now on when we say

$$v = 1/3 \xrightarrow{\text{we mean}} v = \tfrac{1}{3}c$$

That is, we'll use units where $c = 1$. In these units we also have

$$t = 3 \text{ meters} \rightarrow t = \frac{3\,\text{m}}{c} = 10^{-8}\,\text{s}.$$

See Hartle's Appendix A for a nice discussion