General Relativity — Day 8

Feb 17th, 2016

Today

I Last time

II Four Vectors

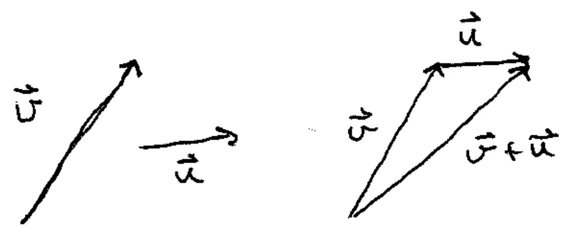
I
- We derived relativity of simultaneity, length contraction, and velocity addition from the Lorentz transformations.
- We decided to adopt units where $c = 1$, so that

$$v = 2/7 \rightsquigarrow v = \tfrac{2}{7} c$$

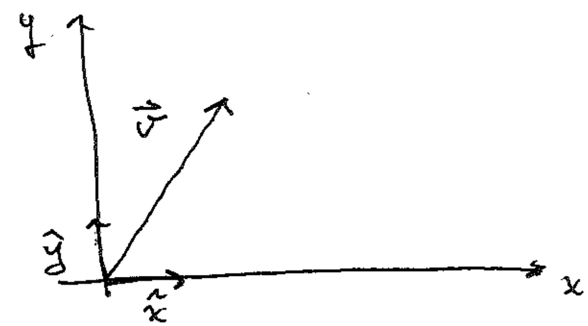
$$t = 3\,m \rightsquigarrow t = \frac{3\,m}{c} = 10^{-8}\,s.$$

II **4-vectors**

A vector is a geometrical object, in particular there is no need to introduce coordinates to discuss vectors:



However, to do calculations it is often nice or convenient to have coordinates.



Along with coordinates, we typically introduce a basis for vectors, e.g. $\{\hat{x}, \hat{y}\}$. Then we write all manner of things

$$\vec{v} = v^x \hat{x} + v^y \hat{y}$$

or

$$\hat{x} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{y} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\vec{v} = \begin{pmatrix} v^x \\ v^y \end{pmatrix}$$

or $v^1 = v^x$ and $v^2 = v^y$ and

$$\vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

or $\{\hat{e}_1, \hat{e}_2\} = \{\hat{x}, \hat{y}\}$ and

$$\vec{v} = \sum_{i=1}^{2} v^i \hat{e}_i = v^1 \hat{e}_1 + v^2 \hat{e}_2 .$$

bold, so I'll use an under twiddle to remind you that something is a 4-vector)

with components

$$x^\mu = (x^0, x^1, x^2, x^3)$$
$$\equiv (t, x, y, z)$$

As the name suggests, these are vectors and:

$$\alpha(\underset{\sim}{a} + \underset{\sim}{b}) = \alpha \underset{\sim}{a} + \alpha \underset{\sim}{b}$$

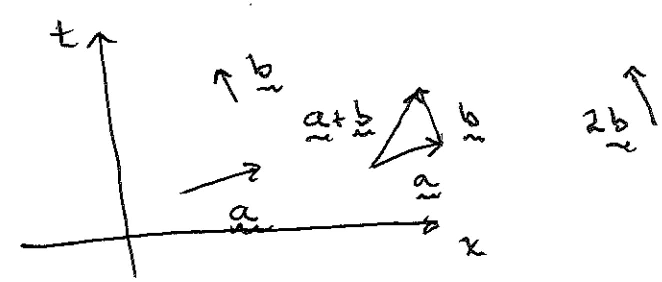
The main property of vectors

is linearity!

$$\alpha(\vec{v} + \vec{u}) = \alpha \vec{v} + \alpha \vec{u}.$$

4-vectors: So far our notation doesn't reflect our recognition of the unification of space and time.

Lorentz transformations suggest that we collect everything into one vector $\underset{\sim}{x}$ (I can't write



For calculations we want to express 4-vectors in a basis; say $\{\underset{\sim}{e}_\mu\}_{\mu=0}^{3}$

$$\underset{\sim}{a} = \sum_{\mu=0}^{3} a^\mu \underset{\sim}{e}_\mu$$
$$= a^0 \underset{\sim}{e}_0 + a^1 \underset{\sim}{e}_1 + a^2 \underset{\sim}{e}_2 + a^3 \underset{\sim}{e}_3$$
$$= a^t \underset{\sim}{e}_t + a^x \underset{\sim}{e}_x + a^y \underset{\sim}{e}_y + a^z \underset{\sim}{e}_z$$

Einstein introduced a wonderful convention: repeated indices — one up and one down — are always summed. This allows us to just write

$$\underset{\sim}{a} = \sum_{\mu=0}^{3} a^{\mu} \underset{\sim}{e}_{\mu} = a^{\mu} \underset{\sim}{e}_{\mu}$$

We also write

$$\underset{\sim}{a} = (a^t, a^i) \quad \text{or} \quad \underset{\sim}{a} = (a^t, \vec{a})$$

<u>Scalar (or dot) product</u>:

Note that the scalar product of our basis vectors is enough to determine all scalar products, because,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a^{\mu} \underset{\sim}{e}_{\mu}) \cdot (b^{\nu} \underset{\sim}{e}_{\nu})$$

$$= a^{\mu} b^{\nu} \underset{\sim}{e}_{\mu} \cdot \underset{\sim}{e}_{\nu}$$

$$\underbrace{\equiv}_{\text{defines } \eta_{\mu\nu}} a^{\mu} b^{\nu} \eta_{\mu\nu}$$

So that

$$\eta_{\mu\nu} \equiv \underset{\sim}{e}_{\mu} \cdot \underset{\sim}{e}_{\nu}.$$

To find $\eta_{\mu\nu}$ we require

$$(\Delta s)^2 = \underset{\sim}{\Delta x} \cdot \underset{\sim}{\Delta x}$$

but then it must be that

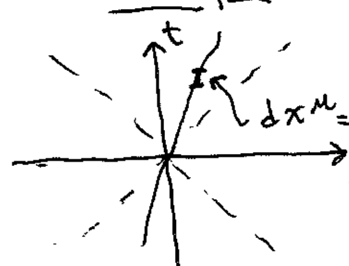
$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In other words

$$ds^2 = \eta_{\mu\nu} \, dx^{\mu} dx^{\nu}$$

and thought of this way, we call $\eta_{\mu\nu}$ the metric of flat spacetime (or the Minkowski metric).

<u>Example</u>: $ds^2 = \eta_{\mu\nu} \, dx^{\mu} dx^{\nu}$



$$= -dt^2 + dx^2 + dy^2 + dz^2$$

$$= -dt^2 + \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right) dt^2$$

$$= -dt^2 \left(1 - v^2\right)$$

$$\Rightarrow \quad ds^2 = -\frac{dt^2}{\gamma^2}$$

But, recall $d\tau = dt/\gamma$, so

$$d\tau^2 = -ds^2 \quad \left(\begin{array}{c} \text{or in our old units} \\ = -ds^2/c^2. \end{array} \right)$$

along a timelike curve.

The dot product in general gives

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$= -a^t b^t + \vec{a} \cdot \vec{b}$$

for flat spacetime.

So, $\underset{\sim}{a}$ is a 4-vector if

$$a^{0'} = \gamma(a^0 - v a^1)$$

$$a^{1'} = \gamma(a^1 - v a^0)$$

$$a^{2'} = a^2$$

$$a^{3'} = a^3.$$

Four vectors are going to allow us to express special relativistic kinematics and dynamics very cleanly.

<u>Note</u>: Physicists often think of a 4-vector as anything that transforms under Lorentz transformations in just the same way that x^μ does.