General Relativity
Day 9

Feb. 19th, 2016

Today's Outline

I S.R. Kinematics

A note we didn't get to last time: Physicists often think of a 4-vector as anything that transforms under Lorentz transformations in just the same way that x^μ does.

Kinematics:
Parametrize a worldline

$$x^\alpha = x^\alpha(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$$

e.g. by proper time. Then,

Proper Velocity:

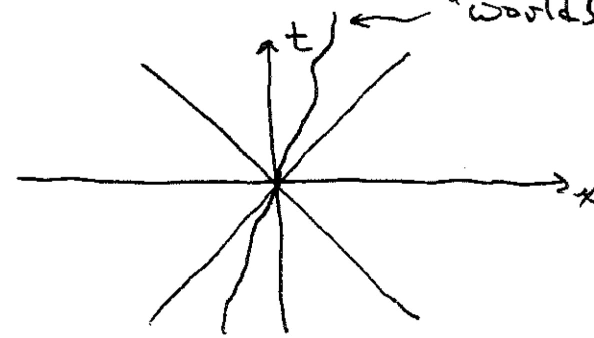
$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

why choose proper time?

$$u^t = \frac{dt}{d\tau} = \gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$$

I S.R. Kinematics & Dynamics

Real particles follow timelike trajectories called worldlines



$$u^x = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma v^x = \frac{v^x}{\sqrt{1-\vec{v}^2}}$$

etc.

So,

$$\boxed{u^\alpha = (\gamma, \gamma\vec{v})}$$

Important calculation,

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = \eta_{\alpha\beta}u^\alpha u^\beta = -\gamma^2 + \gamma^2\vec{v}^2$$

$$= -\frac{(1-\vec{v}^2)}{(1-\vec{v}^2)} = -1.$$

Proper Acceleration: (Newton: $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$)

$$\boxed{a^\alpha = \frac{du^\alpha}{d\tau}}$$

Another important calculation:

$$\frac{d}{d\tau}(\underset{\sim}{u}\cdot\underset{\sim}{u}) = \frac{d}{d\tau}(\eta_{\alpha\beta}u^\alpha u^\beta)$$

$$= \eta_{\alpha\beta}a^\alpha u^\beta + \eta_{\alpha\beta}u^\alpha a^\beta$$

$$= \underset{\sim}{a}\cdot\underset{\sim}{u} + \underset{\sim}{u}\cdot\underset{\sim}{a} = 2\underset{\sim}{a}\cdot\underset{\sim}{u}$$

but also,

$$\frac{d}{d\tau}(\underset{\sim}{u}\cdot\underset{\sim}{u}) = \frac{d}{d\tau}(-1) = 0$$

$$\Rightarrow \boxed{\underset{\sim}{a}\cdot\underset{\sim}{u} = 0}\ !$$

(Four)-acceleration is spacetime orthogonal to (four-) velocity!

Energy & Momentum:

Natural guess is correct:

$$\boxed{p^\alpha = m u^\alpha}$$

Energy-momentum four-vector

$$p^\alpha = (E, \vec{p})$$

Relativistic Momentum: $\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}$

Important calculation number three

$$\underset{\sim}{p}^2 = \underset{\sim}{p}\cdot\underset{\sim}{p} = m^2\underset{\sim}{u}\cdot\underset{\sim}{u} = -m^2$$

Relativistic Energy:

$$E = mu^0 = m\gamma = \frac{m}{\sqrt{1-\vec{v}^2}}$$

$$\approx m\left(1+\frac{1}{2}\vec{v}^2+\cdots\right)$$

$$= m + \frac{1}{2}m\vec{v}^2 + \cdots$$

Note Also: $\frac{\vec{p}}{E} = \frac{\gamma m\vec{v}}{\gamma m} = \vec{v}$ useful!

But then,

$$\underset{\sim}{p}\cdot\underset{\sim}{p} = \eta_{\alpha\beta}p^\alpha p^\beta = -E^2 + \vec{p}^2$$

$$\Rightarrow \boxed{E = (m^2 + p^2)^{1/2}}$$