General Relativity 9
Problem Solving

April 21st, 2016

I Problems

II Curvature begins

I Write out a coordinate basis for vectors on the sphere:

$$\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta = g_{\alpha\beta}$$

or in this 2D setting

$$\underset{\sim}{e}_A \cdot \underset{\sim}{e}_B = g_{AB}$$

with $g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$ for unit radius sphere

Then

$$(\underset{\sim}{e}_\theta)^A = (1, 0)$$

$$(\underset{\sim}{e}_\phi)^A = (0, 1)$$

works.

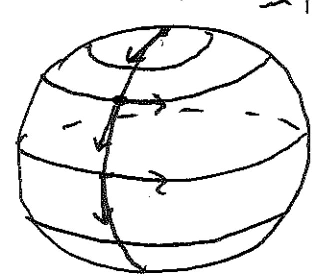
What are these two vectors in partial derivative notation:

$$\underset{\sim}{e}_A = \frac{\partial}{\partial x^A}$$

or

$$(\underset{\sim}{e}_\theta)^A = \frac{\partial}{\partial \theta} \quad \text{and} \quad (\underset{\sim}{e}_\phi)^A = \frac{\partial}{\partial \phi}$$

Draw a few of these basis vectors on the sphere:



← $\underset{\sim}{e}_\phi$ has vanishing length at the poles.

← Both have unit length on the equator.

$$\underset{\sim}{e}_\theta \cdot \underset{\sim}{e}_\theta = (\underset{\sim}{e}_\theta)^A (\underset{\sim}{e}_\theta)^B g_{AB} = 1 \cdot 1 \cdot g_{11} = 1 \checkmark$$

$$\underset{\sim}{e}_\phi \cdot \underset{\sim}{e}_\phi = (\underset{\sim}{e}_\phi)^A (\underset{\sim}{e}_\phi)^B g_{AB} = \sin^2\theta$$

Recall that on the homework we showed:

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta \qquad \left(\begin{array}{c}\text{all other}\\ \text{components}\\ \text{are zero}\end{array}\right)$$

and
$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

What is the covariant derivative of a general vector

$$\underset{\sim}{v} = v^{\theta}\frac{\partial}{\partial\theta} + v^{\phi}\frac{\partial}{\partial\phi} \; ?$$

Well,
$$\nabla_A v^B = \frac{\partial v^B}{\partial x^A} + \Gamma^B_{AC}\, v^C$$

specifically if

$$\underset{\sim}{v} = \cos\phi \frac{\partial}{\partial\phi}$$

then

$$\nabla_\theta v^\theta = 0$$

$$\nabla_\phi v^\theta = -\sin\theta\cos\theta\cos\phi$$

$$\nabla_\theta v^\phi = \cot\theta\cos\phi$$

$$\nabla_\phi v^\phi = -\sin\phi$$

Finally picking a point these can be completely evaluated to numbers.

and so

$$\nabla_\theta v^\theta = \frac{\partial v^\theta}{\partial\theta} + \cancelto{0}{\Gamma^{\theta}_{\theta C}}\, v^C = \frac{\partial v^\theta}{\partial\theta}$$

with

$$\nabla_\phi v^\theta = \frac{\partial v^\theta}{\partial\phi} + \Gamma^{\theta}_{\phi\phi}\, v^\phi$$

$$= \frac{\partial v^\theta}{\partial\phi} - \sin\theta\cos\theta\, v^\phi$$

and

$$\nabla_\theta v^\phi = \frac{\partial v^\phi}{\partial\theta} + \Gamma^{\phi}_{\theta\phi}\, v^\phi = \frac{\partial v^\phi}{\partial\theta} + \cot\theta\, v^\phi$$

finally,

$$\nabla_\phi v^\phi = \frac{\partial v^\phi}{\partial\phi} + \Gamma^{\phi}_{\phi\theta}\, v^\theta = \frac{\partial v^\phi}{\partial\phi} + \cot\theta\, v^\theta.$$

II Curvature begins

On the very first day I argued that Einstein's eqn. took the form

$$\boxed{???} = G\, T^{\mu\nu}$$

Some measure of curvature of spacetime

source = energy-momentum stress tensor

We need a theory of curvature...

Theory of curvature:

(1) Circle:

what is the curvature of this circle?

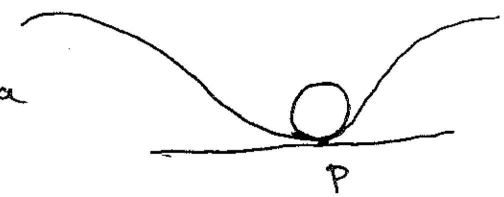
k (the "curvature") is

$$\boxed{k = \frac{1}{R}}$$

compare

to

(2) A curve

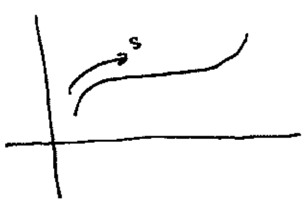


How to define curvature of a curve at the point P? Draw the osculating (best fit) circle at P.

$$\boxed{k = \frac{1}{R|_P}}$$

and as a vector it points towards the center of the osculating circle.

(3) How to calculate $\vec{k}$

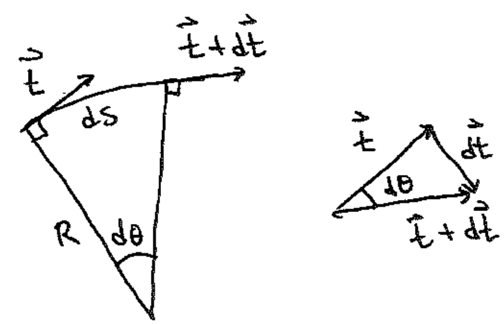


As usual let s be distance along a curve and let $\vec{r}(s)$ define the curve.

Then, $\vec{t} = \frac{d\vec{r}}{ds}$ is the unit tangent vector

and we can define
$$\vec{k} = \frac{d\vec{t}}{ds} = \frac{d^2\vec{r}}{ds^2}$$
as the curvature.

This is sensible,



$d\vec{t}$ points toward the center of the circle $\Rightarrow \hat{k} = d\hat{t}$ and

$$\frac{dt}{t} = d\theta = \frac{dS}{R} \Rightarrow \frac{dt}{dS} = \frac{1}{R} = |\vec{k}|.$$

Work this out for a circle!