Math 141 Name: SOLUTIONS

Exam 2

1. **[15 points (5 pts each)]** Evaluate each of the following derivatives.

(a) Find $\frac{dy}{dx}$ if $y = \tan^{-1}(e^x)$.

$$\frac{dy}{dx} = \frac{1}{1+(e^x)^2}(e^x) = \boxed{\frac{e^x}{1+e^{2x}}}$$

(b) Find $f'(x)$ if $f(x) = \frac{1}{\sqrt{\sin(x^3)}}$.

$$f(x) = [\sin(x^3)]^{-1/2}$$

$$f'(x) = -\frac{1}{2}[\sin(x^3)]^{-3/2}\cos(x^3)\,3x^2$$

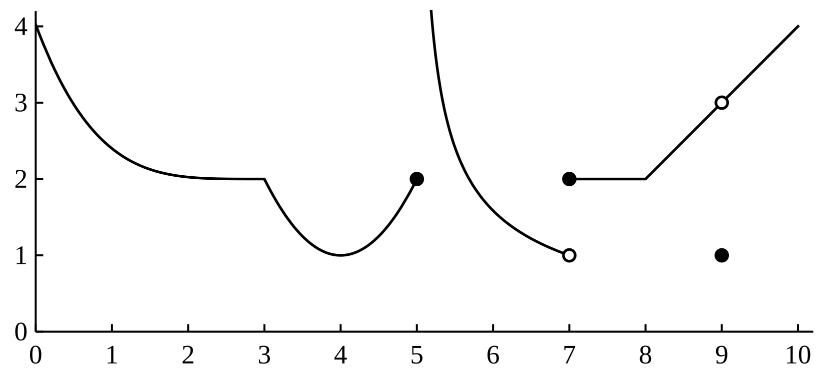
$$= \boxed{-\frac{3x^2\cos(x^3)}{2\sqrt{\sin^3(x^3)}}}$$

(c) Find $g'(x)$ if $g(x) = x^2\ln(1+x^3)$.

$$2x\ln(1+x^3) + x^2\frac{1}{1+x^3}(3x^2)$$

$$= \boxed{2x\ln(1+x^3) + \frac{3x^4}{1+x^3}}$$

2. **[13 points]** The graph of a function $f(x)$ is shown below.



(a) **[5 pts]** List the values of x at which $f(x)$ is not continuous.

$x = 5$, $x = 7$, and $x = 9$

(b) **[4 pts]** What is $\lim_{x \to 7^-} f(x)$?

$\boxed{1}$

(c) **[4 pts]** What is $\lim_{x \to 3^-} f'(x)$? (The left-hand limit of the **derivative**.)

$\boxed{0}$

3. **[4 points]** Suppose that $f(x)$ is an exponential function. Given that $f(0) = 50$ and $f'(0) = 20$, find a formula for $f(x)$.

$$f(x) = 50e^{kx}$$

$$f'(x) = 50ke^{kx}$$

$$f'(0) = 50k$$

$$20 = 50k$$

$$k = 0.4$$

$$\boxed{f(x) = 50e^{0.4x}}$$

4. **[5 points]** Suppose that

$$x^3 + y^3 + 4y = 2.$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y.

$$3x^2 + 3y^2\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

$$(3y^2 + 4)\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \boxed{-\frac{3x^2}{3y^2+4}}$$

5. **[5 points]** Suppose that $f'(x) = 2x^3 - 3x^2 + 6x$ and $f(1) = 2$. What is $f(x)$?

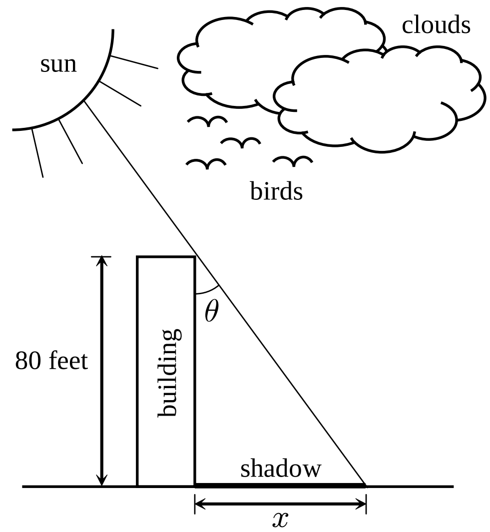
$$f(x) = \frac{1}{2}x^4 - x^3 + 3x^2 + C$$

$$f(1) = \frac{1}{2} - 1 + 3 + C = \frac{5}{2} + C$$

$$\frac{5}{2} + C = 2, \text{ so } C = -\frac{1}{2}$$

$$f(x) = \boxed{\frac{1}{2}x^4 - x^3 + 3x^2 - \frac{1}{2}}$$

6. **[14 points]** One morning at exactly 10:00 AM, the shadow of an 80-foot-tall building is 60 feet long. Because the sun is rising, the shadow is slowly becoming shorter over time.



(a) **[4 points]** Find a formula for the length x of the shadow as a function of the angle θ shown in the picture to the right.

$$\tan\theta = \frac{x}{80}$$

$$\boxed{x = 80\tan\theta}$$

(b) **[4 points]** Take the derivative of your formula to find an equation for $\frac{dx}{dt}$ in terms of θ and $\frac{d\theta}{dt}$.

$$\frac{dx}{dt} = 80\sec^2\theta\,\frac{d\theta}{dt}$$

(c) **[6 points]** Because the sun is rising, the angle θ is decreasing at a rate of 0.25 degrees/minute. How quickly is the length of the shadow decreasing at 10:00 AM?

$$\frac{d\theta}{dt} = -0.25°/\text{min} = -0.00436\ \text{rad}/\text{min}$$

<u>At 10:00 AM</u>

$x = 60$,

so $\tan\theta = 0.75$

so $\theta = 0.6435$ rad

$$\frac{dx}{dt} = 80\sec^2(0.6435)(-0.00436)$$

$$= -0.545$$

$$\boxed{0.545\ \text{ft}/\text{min}}$$

7. **[15 points]** A 12-oz can of soda is put into a refrigerator to cool. Its temperature in Fahrenheit after t minutes is given by the following formula:

$$T = 38 + 36e^{-0.02t}$$

(a) **[3 points]** What is the initial temperature of the soda?

When $t=0$, $T = 38 + 36 = \boxed{74\,^\circ\mathrm{F}}$

(b) **[4 points]** How quickly is the temperature of the soda initially decreasing?

$$\frac{dT}{dt} = (-0.02)\,36e^{-0.02t}$$

Plug in $t=0$: $\frac{dT}{dt} = (-0.02)(36)$

$\boxed{0.72\,^\circ\mathrm{F}/\mathrm{min}}$

(c) **[4 points]** When will the temperature of the soda reach 45° Fahrenheit?

$38 + 36e^{-0.02t} = 45$

$36e^{-0.02t} = 7$

$e^{-0.02t} = 0.1944$

$-0.02t = \ln(0.1944)$

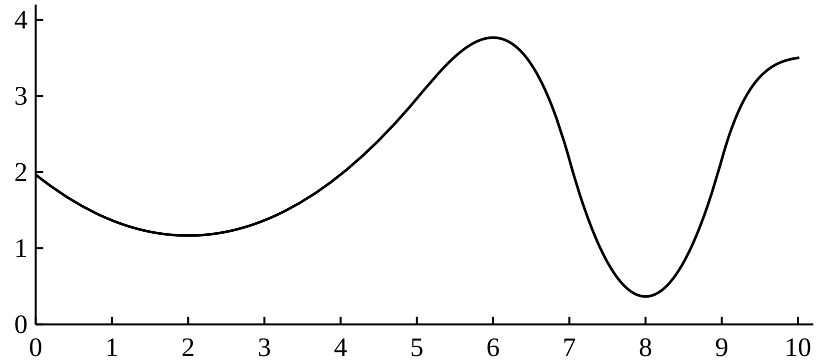
$\boxed{t = 81.88 \text{ min}}$

(d) **[4 points]** What is the temperature inside the refrigerator?

The temperature of the soda approaches the temperature of the refrigerator as $t \to \infty$.

$$\lim_{t\to\infty} 38 + 36e^{-0.02t} = \boxed{38\,^\circ\mathrm{F}}$$

8. **[15 points]** The graph of a function $f(x)$ is shown below.



(a) **[5 pts]** On what interval(s) is $f'(x)$ positive? Explain.

It's positive for $2 < x < 6$ and $8 < x < 10$, since the function is increasing on these intervals.

(b) **[5 pts]** On what interval(s) is $f''(x)$ positive? Explain.

It's positive for $0 < x < 5$ and $7 < x < 9$, since the function is concave up on these intervals.

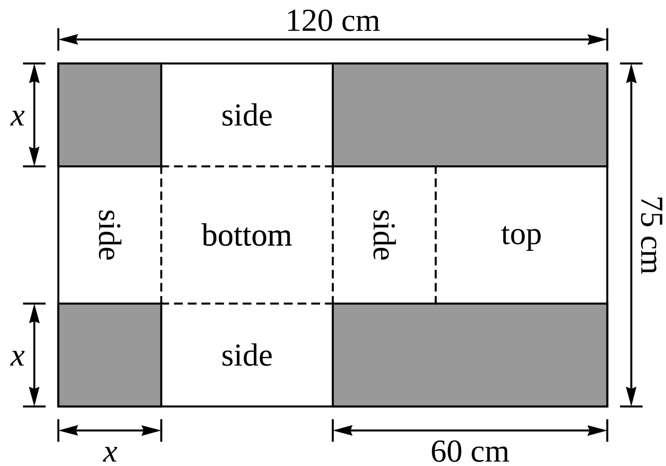
(c) **[5 pts]** Which of the following is **greatest**: $f''(2)$, $f''(4)$, $f''(6)$, or $f''(8)$? Explain.

It's $f''(8)$.

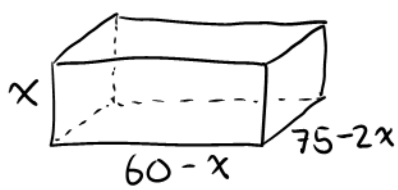
It can't be $f''(6)$, since $f''(6)$ is negative.

$f''(8)$ is greater than $f''(2)$ and $f''(4)$, since the slope is changing more quickly at $x = 8$.

9. **[14 points]** A closed box is to be made from a large (120 cm × 75 cm) sheet of cardboard by cutting out two squares and two rectangles, and then bending the sides and top into position:



(a) **[4 pts]** Find a formula for the volume of the resulting box as a function of the length x.



$$V = x(60-x)(75-2x)$$

(b) **[10 pts]** Take the derivative of your formula from part (a), and use the result to find the value of x that maximizes the volume of the box. (You must show your work to receive full credit.)

$$V = 4500x - 195x^2 + 2x^3$$

$$\frac{dV}{dx} = 4500 - 390x + 6x^2$$

$$6x^2 - 390x + 4500 = 0$$

$$x^2 - 65x + 750 = 0$$

$$(x-15)(x-50) = 0$$

$$x = 15 \quad \text{or} \quad x = 50$$

$$\boxed{x = 15 \text{ cm}}$$

so the box is
15 cm × 45 cm × 45 cm