# Linear Functions

A **linear function** is a function whose graph is a straight line. Such a function can be used to describe variables that change at a constant rate.

For example, the following table shows the accumulation of snow on the morning of a snowstorm:

Time	6:00 am	8:00 am	10:00 am	12:00 pm
Snow depth	2 in.	5 in.	8 in.	11 in.

As you can see, snow was falling at a constant rate of three inches every two hours. If we make a graph of the snow depth as a function of time, the result is a straight line:



The steepness of the line is called its **slope**, and corresponds to the rate at which snow was accumulating. The slope of this line is three inches every two hours, or 1.5 inches/hour.

In general, a linear function involves an independent variable x and a dependent variable y. In the example above, x is the time, and y is the depth of snow. The slope of the line can be obtained from any two data points  $(x_1, y_1)$  and  $(x_2, y_2)$  using the following formula:

slope = 
$$\frac{y_2 - y_1}{x_2 - x_1}$$

The denominator of this formula is a difference of two x values, and the numerator is the difference between the corresponding y values. In the snowfall example above, we could

use any pair of data points to compute the slope. For example, the data points for 6:00 am and 8:00 am give

slope = 
$$\frac{(5 \text{ in.}) - (2 \text{ in.})}{(8:00 \text{ am}) - (6:00 \text{ am})} = \frac{3 \text{ in.}}{2 \text{ hours}} = 1.5 \text{ in./hour.}$$

Similarly, the data points for 8:00 am and 12:00 pm give

slope = 
$$\frac{(11 \text{ in.}) - (5 \text{ in.})}{(12:00 \text{ pm}) - (8:00 \text{ am})} = \frac{6 \text{ in.}}{4 \text{ hours}} = 1.5 \text{ in./hour}$$

#### **Delta Notation**

In calculus, we will often use the Greek letter  $\Delta$  (pronounced "delta") to represent the change in the value of a variable. For example, if *t* represents time, then  $\Delta t$  would represent a change in the value of *t*, i.e. the length of time between two events. This is an example of a **notation**—a convention for writing mathematical formulas.

Using this notation, the formula for slope can be written

slope = 
$$\frac{\Delta y}{\Delta x}$$

where  $\Delta x$  represents a change in the value of x, and  $\Delta y$  represents the corresponding change in the value of y. In particular, given two data points  $(x_1, y_1)$  and  $(x_2, y_2)$ , the quantities  $\Delta x$ and  $\Delta y$  are determined by the formulas

$$\Delta x = x_2 - x_1 \qquad \text{and} \qquad \Delta y = y_2 - y_1$$

### Sign of the Slope

The slope of a linear function can be either positive of negative. In particular, a linear function has positive slope if it is **increasing**, and negative slope if it is **decreasing**. The following example discusses a linear function with negative slope:

**EXAMPLE 1** The data table below shows the total area of old-growth forests in Nigeria from 1995 to 2010:

Year	1995	2000	2005	2010
Forest Cover (km <sup>2</sup> )	149,000	128,000	107,000	86,000

As you can see, Nigeria has been experiencing rapid deforestation over the last 15 years. The rate of deforestation has been roughly constant, with  $21,000 \text{ km}^2$  of forest lost every five years. As a result, the total forest cover has been decreasing linearly. The following graph shows how the data points line up:



Because the total forest cover is decreasing, the corresponding linear function has negative slope. We can calculate the slope using the data points  $x_1 = 2000$  and  $x_2 = 2010$ :

slope = 
$$\frac{(86,000 \text{ km}^2) - (107,000 \text{ km}^2)}{(2010) - (2005)} = \frac{-21,000 \text{ km}^2}{5 \text{ years}} = -4200 \text{ km}^2/\text{years}$$

Note that the numerator is negative, corresponding to the decrease in forest cover between 2005 and 2010. This results in a negative slope, which measures the rate at which the forest cover in Nigeria is decreasing.

By the way, in addition to positive slopes and negative slopes, it is also possible for the slope of a linear function to be zero. This would imply that the value of the function is constant—neither increasing nor decreasing. The graph of such a function is a horizontal line.

#### **Estimating the Slope**

We will often want to be able to estimate the slope of a linear function from its graph. Usually, the best way to do this is to estimate the coordinates of two points on the graph, and then use these to compute the slope. The following example illustrates this process:

**EXAMPLE 2** A pot of water was heated in the microwave for three minutes. The following graph shows the temperature of the water as a function of time:



Estimate the rate at which the temperature of the water was increasing.

SOLUTION This is the same as estimating the slope of the line. We can estimate this slope by estimating the coordinates of two points on the line. To make the estimate as accurate as possible, we will use two points that are as far apart as possible, i.e. the extreme points on the graph.

The left endpoint of the line is at a time of 0:00, and the temperature at this time is approximately 30 °C. The right endpoint is at a time of 3:00, and the temperature at this time is something like 95 °C. Therefore,

slope 
$$\approx \frac{(95 \,^{\circ}\text{C}) - (30 \,^{\circ}\text{C})}{(3:00) - (0:00)} = \frac{65 \,^{\circ}\text{C}}{3 \,\text{min.}} \approx 21.7 \,^{\circ}\text{C/min.}$$

We will often be interested in graphs for which the scales of the horizontal and vertical axes are the same. In this case, it is possible to estimate the slope visually:



For example, a line that slopes upward at an angle of  $45^{\circ}$  would have a slope of 1, as shown above. Keep in mind that this depends on the two axes having the same scale; if scales on the axes are different (as they have been in all of the graphs so far), then this rule does not apply.

## Formulas for Linear Functions

You are probably already familiar with the slope-intercept form for a linear function:

Let f be a linear function with slope m. Then f can be written in the form

f(x) = mx + b,

where b is constant. This is called the **slope-intercept form** for f.

The constant *b* can be interpreted as the value of the function when x = 0:

b = f(0)

It is sometimes called the *y*-intercept for the function, since it corresponds to the height at which the graph of the function intercepts the *y*-axis.

Though the slope-intercept form is simple to remember, it is not always the most useful form for applications. In many applications, it makes no sense to discuss the case where x = 0. For example, if x represents a year, then x = 0 would represent the year zero (also known as 1 BC), which is usually far outside the scope of the discussion.

For this reason, it makes sense to have a form for linear functions based on a different value of *x*:

Let f be a linear function with slope m, and let  $(x_0, y_0)$  be a point on the graph of f. Then

$$f(x) = y_0 + m(x - x_0).$$

This is the **point-slope form** for *f* based at  $(x_0, y_0)$ .

If y = f(x), then the point-slope form can be written

$$y = y_0 + m(x - x_0)$$

We can think of  $x_0$  as a **starting value** or **initial value** for x, and  $y_0$  as the corresponding initial value for y.

**EXAMPLE 3** Consider again our example snowstorm:

Time	6:00 am	8:00 am	10:00 am	12:00 pm
Snow depth	2 in.	5 in.	8 in.	11 in.

Let x be the time in hours, let y be the depth of the snow in inches, and let f be the function relating them. As previously discussed, the function f is linear, with a slope of 1.5 in./hour. Using one of the points from the table, we can obtain a formula for y in terms of x:

$$y = 2 + 1.5(x - 6)$$

You can check that this formula agrees with all of the data points in the table. For example, plugging in 8 for *x* gives us a value of 5 for *y*:

$$2 + 1.5(8 - 6) = 2 + 1.5(2) = 2 + 3 = 5.$$

Also, note that we have omitted the units from the formula; strictly speaking, the 6 represents 6:00 am, the 1.5 should have units of inches/hour, and the 2 should have units of inches.

We can simplify the above formula to obtain the slope-intercept form for *f*:

$$y = 1.5x - 7.$$

However, this form isn't really as useful, since the constant b = -7 has no real meaning. (Theoretically, x = 0 would refer to midnight, but apparently it didn't start snowing until shortly before 6:00 am.)

**EXAMPLE 4** Consider the total area of old-growth forest in Nigeria, as discussed in Example 1:

Year	1995	2000	2005	2010
Forest Cover (km <sup>2</sup> )	149,000	128,000	107,000	86,000

Let *x* be the year, let *y* be the forest cover in square kilometers, and let *f* be the function relating them. As we discussed in Example 1, the function *f* is approximately linear, with a slope of  $-4200 \text{ km}^2/\text{year}$ . Using the year 2000 for  $x_0$ , we obtain a formula for *y* in terms of *x*:

y = 128,000 - 4200(x - 2000)

Before continuing, there are a few aspects of the point-slope formula that we should discuss:

1. We have not explained why the point-slope formula works, but it is actually not that hard to understand:

$$y = y_0 + \overbrace{m(x - x_0)}^{\Delta y}$$

As indicated above, the expression  $x - x_0$  represents a change in the value of x. Multiplying by m gives the corresponding change in the value of y, and then adding this to  $y_0$  gives the final value of y.

2. The point-slope formula can also be viewed as a rearrangement of the formula for slope. Starting with the equation

$$m=\frac{y-y_0}{x-x_0},$$

we can multiply through by the denominator to get

$$y-y_0 = m(x-x_0).$$

Adding  $y_0$  to both sides yields the point-slope formula:

$$y = y_0 + m(x - x_0).$$

3. Finally, the point-slope formula can be thought of as a generalization of the slopeintercept formula. In particular, if we use  $x_0 = 0$ , then  $y_0$  is the same thing as *b*. Plugging these into the point-slope formula gives us

$$y = b + m(x - 0)$$

which simplifies to y = mx + b.

## **Linear Approximation**

As we have seen, linear functions can be used to describe variables that change at a constant rate. In reality, though, most functions are **nonlinear**, meaning that their graphs are curved. For example, if we were to actually measure the accumulation of snow during a snowstorm, we would expect the data to look something like this:

Time	6:00 am	8:00 am	10:00 am	12:00 pm
Snow depth	2.0 in.	4.2 in.	8.6 in.	10.8 in.

As you can see, the rate of snowfall for this snowstorm is changing. If we graph the depth of snow, the result is not a straight line:



Though it may seem that linear functions would useless in this situation, they are actually very helpful for analyzing this snowstorm. The reason is simple: though the graph is curved, certain sections of it are reasonably straight, and we can use linear functions to describe these sections.

For example, consider the portion of the snowstorm between 8:00 am and 10:00 am. As you can see from the graph, snow was falling fairly linearly during this time, so we can approximate the depth of snow using a linear function. We begin by finding the slope:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(8.6 \text{ in.}) - (4.2 \text{ in.})}{(10:00 \text{ am}) - (8:00 \text{ am})} = \frac{4.4 \text{ in.}}{2 \text{ hours}} = 2.2 \text{ in./hour}$$

Though the rate of snowfall may have been changing, this slope still represents the **average rate** of snowfall between 8:00 am and 10:00 am. Using this slope, we can find an approximate formula for the snow depth y as a function of the time x:

$$y \approx 4.2 + 2.2(x - 8)$$

A formula like this is called a **linear approximation**. The symbol  $\approx$  (a misshapen equals sign) is called "approximately equals", and says that the quantity on the right is pretty close to the quantity on the left.

We can use the linear approximation to answer questions about the depth of the snow. For example, if we want to estimate the depth of the snow at 9:00 am, we simply plug x = 9 into our formula:

$$y \approx 4.2 + 2.2(9 - 8) = 6.4$$

Thus the snow depth at 9:00 am was approximately 6.4 inches.

The following examples illustrate this idea further:

**EXAMPLE 5** In the year 2000, the U.S. Postal Service delivered approximately 103,500 pieces of first-class mail. By the year 2010, that number had dropped to 78,500.

- (a) Find an approximate linear formula for the amount of mail delivered each year as a function of time.
- (b) Estimate the amount of mail the USPS will deliver in the year 2012.
- (c) At this rate, what is the last year that the USPS will deliver first-class mail?

SOLUTION With only two data points, there is no particular reason to believe that the volume of mail delivered is decreasing linearly. However, we can make a linear approximation, and it may give us some insight into the mail service. We begin by finding the slope:

average slope = 
$$\frac{78,500 - 103,500}{2010 - 2000} = \frac{-25,000}{10 \text{ years}} = -2500/\text{years}$$

Let *x* be the year, and let *y* be the volume of mail delivered. Then we can write an approximate linear formula for *y* in terms of *x*:

$$y \approx 78,500 - 2500(x - 2010).$$

This is the equation for the line passing through the two data points we were given, and is the answer to part (a).

For part (b), we plug in 2012 for *x*:

$$y \approx 78,500 - 2500(2012 - 2010) = 73,500$$

Thus, the USPS is likely to deliver 73,500 pieces of mail in 2012.

Finally, if we want to estimate when the USPS will stop delivering mail, we can plug in 0 for *y*:

$$0 \approx 78,500 - 2500(x - 2010)$$

Solving this equation for x gives  $x \approx 2041.4$ . Thus, the linear approximation predicts that the U.S. Postal Service will cease operations in the year 2041.

Of course, we have no idea how accurate any of these predictions are. The linear approximation we have made is very simplistic: we are assuming that whatever factors are causing the volume of mail to drop will continue at the same rate. Indeed, all of our predictions are based on only two pieces of data. However, this example does give a sense of some of the math involved in forecasting future trends for large organizations.

**EXAMPLE 6** Let f be a function, and suppose that f(2) = 5 and f(7) = 13.

(a) Find an approximate linear formula for f(x).

(b) Estimate f(4).

(c) Estimate the value of x for which f(x) = 20.

SOLUTION This example is very similar to the last one, except that it is abstract instead of a word problem. The idea is that we have two data points for an unknown function f, and we wish to approximate f using a linear function.

We begin by find the average slope of *f* between x = 2 and x = 7:

average slope 
$$= \frac{13-5}{7-2} = \frac{8}{5} = 1.6.$$

Using  $x_0 = 2$  for the point-slope formula, we obtain

 $f(x) \approx 5 + 1.6(x - 2)$ 

This answers part (a). For part (b), we simply plug in x = 4:

$$f(4) \approx 5 + 1.6(4 - 2) = 8.2.$$

For part (c), we set f(x) equal to 20 and solve for x, The equation we get is

$$20 \approx 5 + 1.6(x - 2)$$

and solving for x gives  $x \approx 11.375$ .