Practice Problem Solutions: Exam 2

1. (a) We know the formula will be of the form $y = be^{kt}$. Since the inital amount is 100-mg, we have that $b = 100$. We can solve for k by plugging in $y = 99.5$ and $t = 1$:

$$99.5 = 100e^{k(1)}$$

Then, $0.995 = e^k$, so $k = \ln(0.995) \approx -0.005012542$. Thus the formula is $\boxed{y = 100e^{-0.005012542t}}$.

(b) We set $y = 50$ in the above formula:

$$50 = 100e^{-0.005012542t}$$

Then, $0.5 = e^{-0.005012542t}$, so $\ln(0.5) = -0.005012542t$. Thus, $t \approx \boxed{138 \text{ days}}$.

2. (a) We know that the formula is of the form $y = be^{kt}$ with $b = 500$. We take the derivative:

$$\frac{dy}{dt} = 500e^{kt}(k)$$

Plugging in $t = 0$ and $\dfrac{dy}{dt} = 150$, we have $150 = 500k$. Thus, $k = 0.3$, so the formula is $\boxed{y = 500e^{0.3t}}$.

(b) We let $y = 7.5$ in the above formula: $y = 500e^{0.3(7.5)} \approx 4743.87$. At 7:30pm, there are approximately $\boxed{4744 \text{ bacteria}}$.

(c) We let $y = 15{,}000$ in the formula from part (a): $15{,}000 = 500e^{0.3t}$. Then, $30 = e^{0.3t}$, so that $\ln(30) = 0.3t$, so $t = \ln(30)/0.3 \approx 11.337$ hours. Thus, there are 15,000 bacteria 11.337 hours after noon, which corresponds to approximately $\boxed{11\text{:}20\text{pm}}$.

3. (a) We have that $\tan\theta = \dfrac{x}{3}$. Thus, $\boxed{x = 3\tan\theta}$.

(b) Taking the derivative of part (a), we get $\boxed{\dfrac{dx}{dt} = 3(1+\tan^2\theta)\dfrac{d\theta}{dt}}$.

(c) We have that $\theta = 30° = \dfrac{\pi}{6}$ radians. Also, since the beam of light rotates four times every minute, we have that $\dfrac{d\theta}{dt} = \dfrac{4(2\pi) \text{ radians}}{1 \text{ minute}} = 8\pi$ radians/minute. We can plug this into the formula from part (b):

$$\frac{dx}{dt} = 3\left(1+\tan^2(\pi/6)\right)(8\pi) \approx \boxed{100.53 \text{ km/min}}$$

4. (a) By implicit differentiation:

$$3x^2 + 3y^2\frac{dy}{dx} = 9\frac{dy}{dx}$$

We take both terms involving $\frac{dy}{dx}$ to the same side: $3x^2 = 9\frac{dy}{dx} - 3y^2\frac{dy}{dx}$. Thus, $3x^2 = \frac{dy}{dx}(9-3y^2)$. Dividing by $9-3y^2$, we get

$$\boxed{\frac{dy}{dx} = \frac{3x^2}{9-3y^2}}$$

(b) We can plug in $x = 2$ and $y = 1$ into the above formula to find the slope at the point $(2,1)$:

$$\frac{dy}{dx} = \frac{3(2)^2}{9-3(1)^2} = \boxed{2}$$

5. (a) We have $y = \frac{3x-5}{x^2-4}$. Using the quotient rule, we get

$$\frac{dy}{dx} = \frac{(x^2-4)(3)-(3x-5)(2x)}{(x^2-4)^2}$$

This simplifies to $\boxed{\frac{dy}{dx} = \frac{-3x^2+10x-12}{(x^2-4)^2}}$.

(b) Since $f(x) = (\ln x - \ln 2)^2$, we have that $f'(x) = 2(\ln x - \ln 2)(1/x)$. Thus,

$$\boxed{f'(x) = \frac{2\ln x - 2\ln 2}{x}}$$

(c) Using the product rule, we have:

$$\frac{d}{dx}\left(x^2e^{-3x}\right) = x^2e^{-3x}(-3) + e^{-3x}(2x) = \boxed{-3x^2e^{-3x} + 2xe^{-3x}}$$

(d) We have $y = x\sec(x^2+1)$. From the formula sheet, we see that $\frac{d}{dx}(\sec x) = \sec x \tan x$. Thus,

$$\frac{dy}{dx} = x\sec(x^2+1)\tan(x^2+1)(2x) + \sec(x^2+1)$$

Simplifying, we have $\boxed{\frac{dy}{dx} = 2x^2\sec(x^2+1)\tan(x^2+1) + \sec(x^2+1)}$.

(e) We have $g(u) = \sin^{-1}(3x^2)$. From the formula sheet, we see that $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$. Thus,

$$g'(u) = \frac{1}{\sqrt{1-(3x^2)^2}}(6x)$$

Simplifying, we have $\boxed{g'(u) = \frac{6x}{\sqrt{1-9x^4}}}$.

(f) We have $y=\tan^3(x^2+1)$. From the formula sheet, we see that $\frac{d}{dx}(\tan x)=\sec^2 x$. Thus,

$$\frac{dy}{dx}=3\tan^2(x^2+1)\sec^2(x^2+1)(2x)$$

Simplifying, we have $\boxed{\frac{dy}{dx}=6x\tan^2(x^2+1)\sec^2(x^2+1)}$.

(g) We have $\theta=\arctan(\ln t)$. Note that $\arctan x=\tan^{-1}x$. From the formula sheet, we see that $\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}$. Thus,

$$\frac{d\theta}{dt}=\frac{1}{1+(\ln t)^2}\left(\frac{1}{t}\right)$$

Simplifying, we have $\boxed{\frac{d\theta}{dt}=\frac{1}{t(1+(\ln t)^2)}}$.

6. (a) Since $f(0.4)=2.8$ and $f(0.6)=4.2$, we can approximate $f'(0.5)$ as follows:

$$f'(0.5)\approx\frac{4.2-2.8}{0.6-0.4}=\boxed{7}$$

(b) Since $f(0.2)=1.8$ and $f(0.4)=2.8$, we can approximate $f'(0.5)$ as follows:

$$f'(0.3)\approx\frac{2.8-1.8}{0.4-0.2}=\boxed{5}$$

(c) Since $f'(0.3)\approx 5$ and $f'(0.5)\approx 7$ by parts (a) and (b), we can approximate $f''(0.4)$ as follows:

$$f''(0.4)\approx\frac{7-5}{0.5-0.3}=\boxed{10}$$

7. (a) Since $a=4$ inches and $b=5$ inches, we have that $A=10\sin\theta$. Differentiating this formula with respect to t, we get

$$\frac{dA}{dt}=10\cos\theta\frac{d\theta}{dt}$$

If $\theta=0.4$ rad and $\frac{d\theta}{dt}=0.15$ rad/min, then we have

$$\frac{dA}{dt}=10\cos(0.4)(0.15)\approx\boxed{1.382\text{ in}^2\text{/min}}$$

(b) If $A=6$ in^2, then we can use the original formula to find θ: $6=10\sin\theta$. Thus, $\sin\theta=0.6$, so $\theta\approx 0.6435$ radians. Also, we have that $\frac{dA}{dt}=0.5$ in^2/min. Now, we can find $\frac{d\theta}{dt}$:

$$0.5=10\cos(0.6435)\frac{d\theta}{dt}$$

Thus, $\boxed{\frac{d\theta}{dt}=.0625\text{ rad/min}}$.

8. (a) We have $f(x) = x\cos x$. Using the product rule, $f'(x) = -x\sin x + \cos x$. Taking the derivative again, we get $f''(x) = -x\cos x - \sin x - \sin x$. Thus, $\boxed{f''(x) = -x\cos x - 2\sin x}$

(b) We have $y = (3x+1)^{1/2}$. Taking the derivative, we get $\dfrac{dy}{dx} = (1/2)(3x+1)^{-1/2}(3)$. Taking the derivative again, we get $\dfrac{d^2y}{dx^2} = (-1/4)(3x+1)^{-3/2}(3)(3)$. Thus, $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{-9}{4(3x+1)^{3/2}}}$.

(c) We have $y^3 + \sin y = x + \cos x$. By implicit differentiation, we get

$$3y^2\frac{dy}{dx} + \cos y\frac{dy}{dx} = 1 - \sin x$$

Thus, $\dfrac{dy}{dx}\left(3y^2 + \cos y\right) = 1 - \sin x$. Dividing by $3y^2 + \cos y$, we get $\boxed{\dfrac{dy}{dx} = \dfrac{1-\sin x}{3y^2 + \cos y}}$

9. (a) First, we take the derivative. Since $f(x) = 36x^{-1} + 25x$, we have $f'(x) = -36x^{-2} + 25$. To find the critical points, we set this equal to zero and solve.

$$-36x^{-2} + 25 = 0 \quad \Rightarrow \quad 25 = \frac{36}{x^2} \quad \Rightarrow \quad 25x^2 = 36 \quad \Rightarrow \quad x = \pm 6/5$$

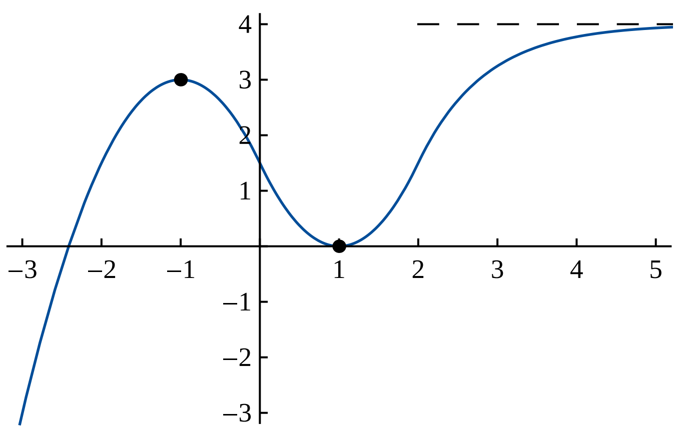
Since $-6/5$ is outside the interval $[0.5, 2.5]$, we can ignore it. Thus, we have one critical point $x = 6/5$ inside the interval. We need to check the critical point and the endpoints:

$$\begin{aligned} f(6/5) &= 60 \\ f(0.5) &= 84.5 \\ f(2.5) &= 76.9 \end{aligned}$$

Thus, the absolute minimum is $\boxed{60}$ and it occurs at $x = 6/5$.

(b) From part (a), we see that the absolute maximum is $\boxed{84.5}$ and it occurs at $x = 0.5$.

10. From the first derivative information, we see that f is increasing for $x < -1$ and $x > 1$ and that f is decreasing for $-1 < x < 1$. From the second derivative information, we see that f is concave down for $x < 0$ and $x > 2$ and that f is decreasing for $0 < x < 2$. Also, since $\lim_{x\to\infty} f(x) = 4$, the graph has a horizontal asymptote at $y = 4$. The graph of f looks like:



11. (a) $\lim_{x\to 3^-} f(x) = \boxed{2}$

(b) $\lim_{x\to\infty} f(x) = \boxed{0}$

(c) $\lim_{x\to 7^+} f(x) = \boxed{-\infty}$

(d) It is not continuous at $\boxed{x = 3 \text{ and } x = 7}$.

(e) Since the graph is concave down at $x = 8$, $f''(8)$ is $\boxed{\text{negative}}$.

(f) Near $x = -2$, the graph is linear, so $\boxed{f''(-2) = 0}$.

12. (a) For $x = -3$, we use the second piece and get that $f(-3) = 1-(-3)^2 = \boxed{-8}$.

(b) Since $x \to -1$ from the right, we use the third piece and get that $\lim_{x\to -1^+} f(x) = 1/(-1) = \boxed{-1}$.

(c) Since $x \to -1$ from the left, we use the second piece and get that $\lim_{x\to -1^-} f(x) = 1-(-1)^2 = \boxed{0}$.

(d) First, we compute the left- and right-hand limits:

$$\begin{aligned}\lim_{x\to -3^+} f(x) &= 1-(-3)^2 = -8\\ \lim_{x\to -3^-} f(x) &= 3(-3)+1 = -8\end{aligned}$$

Since the left- and right-hand limits are equal, we have that $\lim_{x\to -3} f(x) = \boxed{-8}$.

(e) First, we compute the left- and right-hand limits:

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= 5-1 = 4\\ \lim_{x\to 1^-} f(x) &= 1/1 = 1\end{aligned}$$

Since the left- and right-hand limits are not equal, we have that $\lim_{x\to 1} f(x)$ $\boxed{\text{does not exist}}$.

(f) First, we notice that the third piece has a vertical asymptote at $x = 0$, so the function is discontinuous at $x = 0$. The other possible places for the function to be discontinuous would be at $x = -3$, $x = -1$, or $x = 1$.

From part (d), we see that the left- and right-hand limits are equal at $x = -3$. Also, these limits are equal to $f(-3)$, so f is continuous at $x = -3$.

From parts (b) and (c), we see that the left- and right-hand limits are different at $x = -1$, so f is not continuous at $x = -1$.

Finally, from part (e), we see that the left- and right-hand limits are different at $x = 1$, so f is not continuous at $x = 1$. Thus, we have that f is not continuous at $\boxed{x = -1, x = 0, \text{ and } x = 1}$.

13. (a) Plugging in 3.01, 3.001, and 3.0001 for x gives -33.1115, -333.1111, and -3333.1111, respectively. Thus, $\lim_{x\to 3^+} \dfrac{x-5}{x^2-9} = \boxed{-\infty}$.

(b) The graph of $\ln x$ has a vertical asymptote at $x = 0$ with $\ln x \to -\infty$ as $x \to 0^+$. The graph of $\ln(x-2)$ is the graph of $\ln x$ shifted 2 to the right, so the graph of $f(x) = \ln(x-2)$ has a vertical asymptote at $x = 2$. Thus, $\lim_{x\to 2^+} \ln(x-1) = -\infty$.

We can also find the limit numerically. Plugging in 2.0001, 2.00001, and 2.000001 for x gives -9.2103, -11.5129, and -13.8155, respectively. We can see that the limit is $\boxed{-\infty}$.

(c) If $x > -5$, then $\frac{|x+5|}{x+5} = 1$. If $x < -5$, then $\frac{|x+5|}{x+5} = -1$. Thus, we have that

$$\begin{aligned}\lim_{x\to -5^+} \frac{|x+5|}{x+5} &= 1 \\ \lim_{x\to -5^-} \frac{|x+5|}{x+5} &= -1\end{aligned}$$

Since the right-and left-side limits are not equal, we have that $\lim_{x\to -5} \frac{|x+5|}{x+5}$ $\boxed{\text{does not exist}}$.

(d) Plugging in $-.1$ and -0.01 for x gives $-.500626$ and $-.50000625$, respectively. Thus,

$$\lim_{x\to 0^-} \frac{x}{\sqrt{4x^2 - x^4}} = \boxed{-1/2}$$

14. The penny's acceleration is $a = -32$ feet/second^2. We can take the antiderivative to find the velocity: $v = -32t + C_1$. Since $v = 0$ when $t = 0$, we have that $0 = -32(0) + C_1$. Thus, $C_1 = 0$, so the velocity is $v = -32t$.

To find the height h of the penny, we can take the antiderivative of v. We get that $h = -16t^2 + C_2$. The initial height of the penny is 400 ft, so we can plug in $h = 400$ and $t = 0$ to find C_2: $400 = -16(0)^2 + C_2$. Thus, $C_2 = 400$, so the height is $h = -16t^2 + 400$.

We want to know when the penny hits the ground, which is when $h = 0$. So, we plug in $h = 0$ and solve:

$$0 = -16t^2 + 400 \;\Rightarrow\; 16t^2 = 400 \;\Rightarrow\; t = \pm 5$$

Thus, the penny hits the ground $\boxed{\text{5 seconds}}$ after it is dropped.

15. The height of the box is a, the length is $12 - 2a$, and the width is $6 - 2a$. Thus, the volume of the box is

$$V = a(12-2a)(6-2a) = 4a^3 - 36a^2 + 72a$$

We take the derivative: $\frac{dV}{da} = 12a^2 - 72a + 72$. The maximum will occur when the derivative equals 0, so we set the derivative equal to 0 and solve. We have $12a^2 - 72a + 72 = 0$, which simplifies to $a^2 - 6a + 6 = 0$. To solve this equation, we can use the quadratic formula:

$$a = \frac{6 \pm \sqrt{36 - 4(6)}}{2}$$

Thus, $a \approx 4.732$ or $a \approx 1.268$. Note that a is between 0 and 3, so the maximum is at $a \approx 1.268$ and the maximum volume is $V = 1.268(12 - 2\cdot 1.268)(6 - 2\cdot 1.268) = \boxed{41.569 \text{ ft}^3}$.