Practice Problem Solutions: Final Exam

1. The five rectangles will each have width 0.2. The following table shows the leftmost x-coordinate, height, and area of each rectangle:

x-coordinate	Height	Area
0.0	1	0.2
0.2	0.99206	0.19841
0.4	0.93985	0.18797
0.6	0.82237	0.16447
0.8	0.66138	0.13228

Adding up the areas of the five rectangles we get that $\int_0^1 \frac{1}{1+x^3}\,dx \approx \boxed{0.883}$.

2. The surface area is $A = \pi r^2 + 2\pi rh$. We want to minimize the surface area, so we first need to find a formula for the surface area that involves only one variable. The volume of the cylinder is $V = \pi r^2 h$, so we have that $300 = \pi r^2 h$. Solving for h, we have that $h = \frac{300}{\pi r^2}$ We can plug this into the formula for surface area and obtain:

$$A = \pi r^2 + 2\pi r\left(\frac{300}{\pi r^2}\right)$$

Simplifying, we get $A = \pi r^2 + 600/r$. We want to minimize the surface area, so we take the derivative:

$$\frac{dA}{dt} = 2\pi r - \frac{600}{r^2}$$

The minimum will occur when the derivative equals 0. Setting equal to 0, we have $2\pi r - 600/r^2 = 0$. Solving, we find that $r = \sqrt[3]{300/\pi} \approx 4.571$. When $r = 4.571$, we have that $h = \frac{300}{\pi(4.571)^2} \approx 4.571$. Thus, the surface area of the cylinder is minimized when $\boxed{r = 4.571 \text{ cm and } h = 4.571 \text{ cm}}$.

3. (a) $$\int_1^2 x(x+2)\,dx = \int_1^2 (x^2+2x)\,dx = \left[\frac{1}{3}x^3 + x^2\right]_1^2 = \left(\frac{1}{3}(2)^3 + 2^2\right) - \left(\frac{1}{3}(1)^3 + 1^3\right) = \boxed{\frac{16}{3}}$$

(b) $$\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}\left(4^{3/2}\right) - \frac{2}{3}\left(1^{3/2}\right) = \boxed{\frac{14}{3}}$$

(c) $$\int_0^2 \frac{1}{5x+1}\,dx = \left[\frac{1}{5}\ln(5x+1)\right]_0^2 = \frac{1}{5}\ln(11) - \frac{1}{5}\ln(1) \approx \boxed{\frac{\ln(11)}{5}}$$

(d) $\displaystyle\int_0^{\pi/2} \cos(3x)\,dx \;=\; \left[\frac{1}{3}\sin(3x)\right]_0^{\pi/2} \;=\; \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) - \frac{1}{3}\sin(0) \;=\; \boxed{-\frac{1}{3}}$

(e) $\displaystyle\int_{-1}^{1} (1-x)^5\,dx \;=\; \left[\frac{-1}{6}(1-x)^6\right]_{-1}^{1} \;=\; \frac{-1}{6}(1-1)^6 + \frac{1}{6}(1+1)^6 \;=\; \boxed{\frac{32}{3}}$

(f) $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx \;=\; \Big[\tan x\Big]_0^{\pi/4} \;=\; \tan\left(\frac{\pi}{4}\right) - \tan(0) \;=\; \boxed{1}$

(g) $\displaystyle\int_0^2 (e^x+1)^2 \;=\; \int_0^2 \left(e^{2x}+2e^x+1\right) \;=\; \left[\frac{1}{2}e^{2x}+2e^x+x\right]_0^2 \;=\; \left(\frac{1}{2}e^4+2e^2+2\right) - \left(\frac{1}{2}e^0+2e^0+0\right)$

$\displaystyle = \frac{e^4}{2}+2e^2-\frac{1}{2} \;\approx\; \boxed{41.577}$

(h) $\displaystyle\int_0^1 \left(x^3\cos x+3x^2\sin x\right)dx \;=\; \Big[x^3\sin x\Big]_0^1 \;=\; \sin(1) - 0 \;=\; \sin(1) \;\approx\; \boxed{0.01745}$

(i) $\displaystyle\int_0^4 2x\sqrt{x^2+9} \;=\; \int_0^4 2x\left(x^2+9\right)^{1/2} \;=\; \left[\frac{2}{3}\left(x^2+9\right)^{3/2}\right]_0^4 \;=\; \frac{2}{3}\left(25^{3/2}\right) - \frac{2}{3}\left(9^{3/2}\right) \;=\; \boxed{\frac{196}{3}}$

(j) $\displaystyle\int_0^1 \frac{e^{2x}}{1+e^{2x}}\,dx \;=\; \left[\frac{1}{2}\ln(1+e^{2x})\right]_0^1 \;=\; \frac{1}{2}\ln(1+e^2) - \frac{1}{2}\ln(2) \;\approx\; \boxed{0.7169}$

4. (a) For each interval of 0.2 seconds, we can estimate the average velocity, and then use that to estimate the distance traveled by the object.

Time Interval	Average Velocity	Distance Traveled
0.0 to 0.2	3.8 m/s	0.76 m
0.2 to 0.4	4.15 m/s	0.83 m
0.4 to 0.6	4.45 m/s	0.89 m
0.6 to 0.8	4.7 m/s	0.94 m
0.8 to 1.0	4.85 m/s	0.97 m

Adding these together, the total distance traveled is approximately $\boxed{\text{4.39 meters.}}$

(b) We can use the velocity at $t = 0.2$ and $t = 0.6$ to estimate the acceleration at 0.4:

$$\text{acceleration} \approx \frac{4.6-4}{0.6-0.2} = \boxed{1.5 \text{ m/s}^2}$$

(c) We can use a linear approximation to estimate the velocity at $t = 0.43$. We'll use the velocities at $t = 0.4$ and $t = 0.6$, and assume that the velocity increases linearly from $t = 0.4$ to $t = 0.6$. The slope of the linear approximation is: $m = \dfrac{4.6-4.3}{0.2} = 1.5$. Thus, the velocity at $t = 0.43$ is approximately:

$$\text{velocity} \approx 4.3 + 1.5(0.43-0.4) = \boxed{4.345 \text{ m/s}}$$

5. By the Pythagorean Theorem, we have that $L^2 = x^2 + y^2$. Taking the derivative, we have that

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

We know that $\frac{dx}{dt} = 15$ miles/hour and $\frac{dy}{dt} = 20$ miles/hour. Also, we know that $x = 80$ miles, $y = 60$ miles. By the Pythagorean Theorem, $L = \sqrt{80^2 + 60^2} = 100$. Plugging these into the equation, we get:

$$2(100)\frac{dL}{dt} = 2(80)(15) + 2(60)(20)$$

Thus, we have that $200\frac{dL}{dt} = 4800$. Thus, $\frac{dL}{dt} = \boxed{24 \text{ miles/hour}}$.

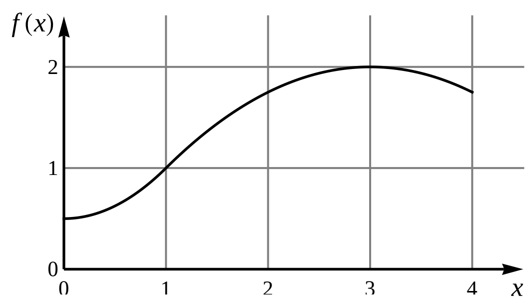
6. (a) Note that $\int_0^1 f'(x)\,dx = 1/2$. Since $f(0) = 1/2$, we have that $f(1) = 1/2 + 1/2 = \boxed{1}$.

   Note that $\int_0^4 f'(x)\,dx = 5/4$. Since $f(0) = 1/2$, we have that $f(4) = 1/2 + 5/4 = \boxed{7/4}$.

   (b) The maximum value of $f(x)$ occurs at $x = 3$. For $x < 3$, the graph of f' is positive, so f is increasing. For $x > 3$, the graph of f' is negative, so f is decreasing. We have that $f(3) = 1/2 + 3/2 = \boxed{2}$.

   (c) The second derivative is positive when the first derivative is increasing. Thus, $\boxed{f''(x) > 0 \text{ for } 0 < x < 1}$. The second derivative is negative when the first derivative is decreasing. Thus, $\boxed{f''(x) < 0 \text{ for } 1 < x < 4.}$

   (d)



7. (a) Since $g(u) = (1+u^2)^{1/2}$, we have that $g'(u) = \frac{1}{2}(1+u^2)^{-1/2}(2u)$. Simplifying, we have that $\boxed{g'(u) = \frac{u}{\sqrt{1+u^2}}}$.

   (b) Since $x = \frac{1}{5}\sec\left(\frac{\theta}{2}\right)$, we have that $\boxed{\frac{dx}{d\theta} = \frac{1}{10}\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)}$.

   (c) Since $y = \ln(\sin x)$, we have that $\boxed{\frac{dy}{dx} = \frac{\cos x}{\sin x}}$.

(d) By the Fundamental Theorem of Calculus, $\boxed{f'(x) = \dfrac{1}{1+x^4}}$.

(e) We have that $4x^2 + y^2 = 9$. We take the derivative with respect to x:

$$8x + 2y\frac{dy}{dx} = 0$$

Then, we solve for $\frac{dy}{dx}$. We get $\boxed{\dfrac{dy}{dx} = \dfrac{-8x}{2y}}$

(f) We have that $f'(x) = x^2 + e^{2x}$. We take the antiderivative to get that $f(x) = \frac{1}{3}x^3 + \frac{1}{2}e^{2x} + C$. We know that $f(0) = 4$, and we can plug this in to find C. We get that $4 = 0 + \frac{1}{2}e^0 + C$, so $C = 7/2$. Thus, $\boxed{f(x) = \dfrac{1}{3}x^3 + \dfrac{1}{2}e^{2x} + \dfrac{7}{2}}$.

8. (a) $f(-2) = 3(-2) - 1 = \boxed{-7}$

(b) $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} x^2 - 2 = (-2)^2 - 2 = \boxed{2}$

(c) $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} 3x - 1 = 3(-2) - 1 = \boxed{-7}$

(d) $\int_1^6 f(x)\,dx \;=\; \int_1^3 (2x-3)\,dx + \int_3^6 (6-x)\,dx \;=\; 2 + 4.5 = \boxed{6.5}$

(e) It is not continuous at $\boxed{x = -2}$, since the limits from the right and left are different. (At the other breakpoints, the limits from the left and right are equal.)

(f) It is not differentiable at $\boxed{x = -2 \text{ and } x = 3}$. At $x = -2$, it is not continuous, so it is not differentiable. At $x = 3$, the derivative on the left is 2, while the derivative on the right is -1. Note that f is differentiable at $x = 1$, where the derivative from the left and right is 2.

9. (a) $\lim_{x\to\infty} \dfrac{x}{x+5} = \boxed{1}$

(b) $\lim_{x\to 1^+} \dfrac{x}{1-x} = \boxed{-\infty}$.

(c) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} = \boxed{0}$.

(d) $\lim_{x\to\pi/2^+} \tan x = \boxed{-\infty}$.

(e) $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = \boxed{1}$

10. First, we need to find the intersection points of the curves $y = x^2 - x$ and $y = x$. At the intersection, we have $x^2 - x = x$. Solving, we have that $x = 0$ or $x = 2$. Thus, to find the area we compute the integral $\int_0^2 \left(x - (x^2 - x)\right)\,dx$. We evaluate the integral:

$$\int_0^2 \left(x-(x^2-x)\right)\,dx = \int_0^2 \left(2x - x^2\right)\,dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = \left(2^2 - \frac{1}{3}(2^3)\right) - 0 = \boxed{4/3}$$

11. (a) First, we need to find the x-coordinates where the curve $y = x - x^3$ intersects the x-axis. We solve $x - x^3 = 0$. This is equivalent to $x(1-x^2) = 0$, so the solutions are $x = 0, -1$, and 1. Based on the picture, we are interested in the area under the curve from $x = 0$ to $x = 1$. So, we want to evaluate the integral $\int_0^1 (x - x^3)\,dx$.

$$\int_0^1 (x-x^3)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \left(\frac{1}{2} - \frac{1}{4}\right) - 0 = \boxed{\frac{1}{4}}$$

(b) When we rotate the region around the x-axis, an infinitesimal slice has volume $dV = \pi R^2\,dx$ where $R = x - x^3$. Thus, the volume is $V = \int_0^1 \pi\left(x - x^3\right)^2\,dx$.

$$\int_0^1 \pi\left(x-x^3\right)^2\,dx = \int_0^1 \pi\left(x^2 - 2x^4 + x^6\right)\,dx = \left[\pi\left(\frac{1}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7\right)\right]_0^1 = \boxed{\frac{8\pi}{105}}$$

12. (a) We know that $\int_0^{3.02} f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^{3.02} f(x)\,dx$, and we know that $\int_0^3 f(x)\,dx = 6$. Thus, we just need to estimate the value of $\int_3^{3.02} f(x)\,dx$. Since $f(3) = 4$, the value of this integral is approximately equal to the area of a rectangle with width 0.02 and height 4. Thus, the value of the integral is approximately $0.02(4) = 0.08$, so

$$\int_0^{3.02} f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^{3.02} f(x)\,dx \approx 6 + 0.08 = \boxed{6.08}$$

(b) Since $\int_0^3 f(x)\,dx = 6$, the value of b will be a little larger than 3. The value of the integral $\int_3^b f(x)\,dx$ is equal to 0.2, and is also approximated a by rectangle with height 4 (since $f(3) = 4$). If the width of the rectangle is Δx, then the area is $4\Delta x = 0.2$. Solving, we get that $\Delta x = 0.05$. Thus, the width of the rectangle is 0.05, so we have $\boxed{b \approx 3.05}$.

13. We can differentiate the equation to find the relationship between $\frac{dK}{dt}$ and $\frac{dv}{dt}$ (note that m is a constant):

$$\frac{dK}{dt} = mv\frac{dv}{dt}$$

We have that $m = 2.00$ kg, $v = 23.0$ m/s, and $\frac{dv}{dt} = 6.00$ m/s^2 (since the acceleration is the derivative of velocity). Plugging these in, we get:

$$\frac{dK}{dt} = (2.00)(23.0)(6.0)276 \text{ kg} \cdot \text{m}^2/\text{s}^3$$

Thus, $\frac{dK}{dt} = \boxed{276 \text{ kg} \cdot \text{m}^2/\text{s}^3}$.

14. The area is approximately a rectangle with width 0.03 and height $\left(1 - 0.4^2\right) - 0.4^2 = 0.68$. Thus, the area is $(0.68)(0.03) = \boxed{0.0204}$.