Takehome Midterm

Math 261, Spring 2013

This midterm has three problems, some of which have several parts. Your solutions must be written up in LaTeX, and are due on Friday, March 22. **Late solutions will not be accepted.**

You must work completely on your own, consulting only the textbook, your course notes, and your homeworks as references. If you have questions, you can come to my office hours or ask me via e-mail. Good luck!

1. An integer a is said to be **square-free** if the following condition holds: for every integer n, if $n^2|a$ then $n = \pm 1$.

   (a) **[5 pts.]** List all square-free integers between 1 and 25.

   (b) **[45 pts.]** Prove or give a counterexample for each of the following statements:

      i. For every integer a, either a is square-free or there exists an integer m such that $a = m^2$.

      ii. Let a and b be integers. If b is square-free and $a|b$, then a is square-free.

      iii. If a and b are integers and ab is square-free, then a and b are relatively prime.

      iv. For every integer a, there exists an integer b so that $a < b$ and b is not square-free.

      v. If k is an integer, then $k^2 + 1$ is square-free.

2. **[20 pts.]** Let A, B, and C be sets, and suppose that $A \subseteq B$ and $B \subseteq C$. Prove that

$$C - A = (C - B) \cup (B - A).$$

3. Let A and B be sets, and let $f\colon A \to B$ be a function.

   (a) **[10 pts.]** If P and Q are subsets of A, prove that $f(P) - f(Q) \subseteq f(P - Q)$.

   (b) **[20 pts.]** If C and D are subsets of B, prove that $f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$.