Practice Problems for Final

Practice Problems from Textbook:

Chapter 4:

§ 4.1 # 1, 3

§ 4.2 # 1 (parts 1, 3)

§ 4.3 # 1, 2

§ 4.4 # 1, 2

Chapter 5:

§ 5.1 # 3

§ 5.2 # 1, 2, 3

§ 5.3 # 1

Chapter 6

§ 6.3 # 1 (parts 3, 4, 6)

Additional Problems:

1. (a) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 3$. What is the range of f? What is $f\left([-1,2]\right)$? What is $f^{-1}\left([-3,4]\right)$?

   (b) Consider the function $g\colon [-1,5] \to \mathbb{R}$ defined by $g(x) = 3x - 5$. What is the range of g? What is $g\left([1,2]\right)$? What is $g^{-1}\left([7,12]\right)$?

   (c) Consider the function $h\colon \mathbb{N} \to \mathbb{N}$ defined by $h(n) = n + 2$. What is the range of h? What is $h\left(\{1,2,3\}\right)$? What is $h^{-1}\left(\{1,3,5,7\}\right)$?

2. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ x+5, & \text{if } x < 0 \end{cases} \qquad g(x) = \begin{cases} 3x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

   Find $f \circ g$ and $g \circ f$.

3. (a) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 5$. Prove that f is bijective.

   (b) Consider the function $g\colon (0,\infty) \to \mathbb{R}$ defined by $g(x) = 1/x$. Prove that g is injective. Is g surjective? Why or why not?

   (c) Consider the function $h\colon \mathbb{R} \to [1,\infty)$ defined by $h(x) = x^2 + 1$. Prove that h is surjective. Is h injective? Why or why not?

4. Solve each of the following equations in the given set $\mathbb{Z}_n$. (In some cases, there may be no solutions.)

   (a) $[5] + x = [3]$ in $\mathbb{Z}_8$

   (b) $[3] \cdot x = [2]$ in $\mathbb{Z}_5$

   (c) $[2] \cdot x = [3]$ in $\mathbb{Z}_4$

   (d) $[3] \cdot x = [6]$ in $\mathbb{Z}_9$

5. Let $\sim$ be the relation on $\mathbb{R}^2 - \{(0,0)\}$ defined by $(x, y) \sim (w, z)$ if and only if there exists $k \in \mathbb{R} - \{0\}$ such that $x = kw$ and $y = kz$. Prove that $\sim$ is an equivalence relation.

6. Let $\sim$ be the relation on $\mathbb{N}$ defined by $a \sim b$ if and only if there exists $n \in \mathbb{Z}$ such that $a = 2^n b$, for all $a, b \in \mathbb{N}$.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Find the equivalence classes [0] and [3].

7. Use induction to prove that the following formulas hold for all $n \in \mathbb{N}$.

   (a) $2 + 5 + 8 + 11 + 14 + \cdots + (3n - 1) = \dfrac{(n)(3n+1)}{2}$

   (b) $2 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1} - 2$