Answers to Practice Problems

Practice Problems from Textbook:

Chapter 4:

4.1.1. (1), (3), and (6) are functions.

4.1.3. (i) and (iv) represent functions.

4.2.1. (1) $[-5, \infty)$

(3) $(3, \infty)$

4.3.1. (1) $(f \circ g)(x) = e^{\sin x}$
$(g \circ f)(x) = \sin(e^x)$

(2) $(f \circ g)(x) = x^{-21}$
$(g \circ f)(x) = x^{-21}$

(3) $(f \circ g)(x) = x$
$(g \circ f)(x) = x$

(4) $(f \circ g)(x) = \lceil x \rceil$
$(g \circ f)(x) = \lfloor x \rfloor$

4.3.2. (1) $g(x) = x + 7$
$h(x) = \sqrt[3]{x}$

(2) $g(x) = \sqrt[3]{x}$
$h(x) = x + 7$

(3) One such pair of functions h and g is:

$$g(x) = \begin{cases} x^3, & \text{if } 0 \leq x \\ x^2, & \text{if } x < 0 \end{cases} \qquad h(x) = x^2$$

(4) One such pair of functions h and g is

$$g(x) = \begin{cases} x^3, & \text{if } 0 \leq x \\ x/2, & \text{if } x < 0 \end{cases} \qquad h(x) = \begin{cases} x, & \text{if } 0 \leq x \\ 2x, & \text{if } x < 0 \end{cases}$$

4.4.1. (1) **Theorem.** *Let $t\colon (1,\infty) \to \mathbb{R}$ be defined by $t(x) = \ln x$ for all $x \in (1,\infty)$. The function t is injective. It is not surjective.*

Proof. First, we will show that t is injective. Let $x, y \in (1,\infty)$. Suppose that $t(x) = t(y)$. Then $\ln(x) = \ln(y)$. Thus, $x = y$, so t is injective.

Note that there is no $x \in (1,\infty)$ such that $t(x) = -1$ since $\ln(x)$ is always positive for $x > 1$. Thus, t is not surjective. □

(2) **Theorem.** *Let $s\colon \mathbb{R} \to \mathbb{R}$ be defined by $s(x) = x^4 - 5$ for all $x \in \mathbb{R}$. The function s is neither injective nor bijective.*

Proof. Note that $s(1) = -4$ and $s(-1) = -4$. Thus, s is not injective. Also, note that there is no $x \in \mathbb{R}$ such that $s(x) = -7$. Thus, s is not surjective. □

(3) **Theorem.** *Let $g\colon [0,\infty) \to [0,1)$ be defined by $g(x) = \frac{x}{1+x}$ for all $x \in [0,\infty)$. The function g is both injective and surjective.*

Proof. First, we will show that g is injective. Let $x, y \in [0,\infty)$. Suppose that $g(x) = g(y)$. Then $\frac{x}{1+x} = \frac{y}{1+y}$. Cross-multiplying, we have that $x(1+y) = y(1+x)$. Thus, $x + xy = y + xy$. Subtracting xy from both sides, we get that $x = y$. Thus, g is injective.

Next, we will show that g is surjective. Let $y \in [0,1)$. Let $x = \frac{y}{1-y}$. (Note that x exists since $0 \leq y < 1$.) Then, $g(x) = g\left(\frac{y}{1-y}\right) = \dfrac{y/(1-y)}{1+y/(1-y)}$. If we multiply the top and bottom of this fraction by $1-y$, we get:

$$g(x) = \frac{y}{1-y+y} = \frac{y}{1} = y$$

Thus, g is surjective. □

(4) **Theorem.** *Let $k\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $k((x,y)) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. The function k is neither injective nor surjective.*

Proof. Note that $k((-1,0)) = 1$ and $k((1,0)) = 1$. Thus, k is not injective. Also, note that there is no $(x,y) \in \mathbb{R}^2$ such that $k((x,y)) = -1$, since $x^2 + y^2$ is always positive. Thus, k is not surjective. □

(5) **Theorem.** *Let* $Q\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ *be defined by* $Q(n) = \{1, 2, \ldots, n\}$ *for all* $n \in \mathbb{N}$. *The function* Q *is injective. It is not surjective.*

Proof. First, we will show that Q is injective. Let $a, b \in \mathbb{N}$. Suppose that $Q(a) = Q(b)$. Then, $\{1, 2, \ldots, a\} = \{1, 2, \ldots, b\}$. Since these two sets are equal, we must have $a \in \{1, 2, \ldots, b\}$ and $b \in \{1, 2, \ldots, a\}$. Thus, it must be the case that $1 \le a \le b$ and $1 \le b \le a$. Since $a \le b$ and $b \le a$, it follows that $a = b$. Thus, Q is injective.

Note that the set $\{2\}$ is in $\mathcal{P}(\mathbb{N})$, but there is no $n \in \mathbb{N}$ such that $Q(n) = \{2\}$. Thus, Q is not surjective. □

4.4.2. (1) Injective, not surjective.
(2) Injective and surjective.
(3) Injective, not surjective.
(4) Injective and surjective.

Chapter 5:

5.1.3. (1) Symmetric, not reflexive and not transitive.
(2) Reflexive and transitive, not symmetric.
(3) Transitive, not reflexive and not symmetric.
(4) Reflexive, symmetric, and transitive.
(5) Symmetric, not reflexive and not transitive.
(6) Not reflexive, not symmetric, and not transitive.
(7) Reflexive and transitive, not symmetric.

5.2.1. (1), (4), and (5) are true. (2) and (3) are false.

5.2.2. (1) $x = [5]$
(2) $x = [9]$
(3) no solutions
(4) $x = [2]$ or $[7]$
(5) $x = [4]$

5.2.3. One example is $n = 5$, $a = 2$, $b = 3$. Then $a^2 = 4$ and $b^2 = 9$, and $4 \equiv 9 \pmod 5$ but $2 \not\equiv 3 \pmod 5$.

5.3.1 (1), (3), (5), (6) are equivalence relations

Chapter 6:

6.3.1 (3) **Theorem.** *Let* $n \in \mathbb{N}$. *Then,* $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Proof. We proceed by induction on n.

Base Case If $n = 1$, then both sides of the equation are equal to 1, so the formula holds in this case.

Induction Step Now suppose that the formula holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\
&= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}
\end{aligned}$$

so the formula holds for $n+1$ as well.

Therefore, by induction, the formula holds for all $n \in \mathbb{N}$. □

(4) **Theorem.** *Let* $n \in \mathbb{N}$. *Then,* $1^3 + 3^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$.

Proof. We proceed by induction on n.

Base Case If $n = 1$, then both sides of the equation are equal to 1, so the formula holds in this case.

Induction Step Now suppose that the formula holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2n-1)^3 + (2(n+1)-1)^3 &= n^2(2n^2-1) + (2(n+1)-1)^3 \\
&= n^2(2n^2-1) + (2n+1)^3 \\
&= 2n^4 - n^2 + (8n^3 + 12n^2 + 6n + 1) \\
&= 2n^4 + 8n^3 + 11n^2 + 6n + 1 \\
&= (n+1)(2n^3 + 6n^2 + 5n + 1) \\
&= (n+1)^2(2n^2 + 4n + 1) \\
&= (n+1)^2 \left(2(n+1)^2 - 1\right)
\end{aligned}$$

so the formula holds for $n+1$ as well.

Therefore, by induction, the formula holds for all $n \in \mathbb{N}$. □

(6) **Theorem.** *Let* $n \in \mathbb{N}$. *Then,* $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

Proof. We proceed by induction on n.

Base Case If $n = 1$, then both sides of the equation are equal to $1/2$, so the formula holds in this case.

Induction Step Now suppose that the formula holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}
\end{aligned}$$

so the formula holds for $n+1$ as well.

Therefore, by induction, the formula holds for all $n \in \mathbb{N}$. $\square$

Additional Problems:

1. (a) The range of f is $[3, \infty)$.
   $f([-1,2]) = [3,7]$
   $f^{-1}([-3,4]) = [-1,1]$

   (b) The range of g is $[-8, 10]$.
   $g([1,2]) = [-2,1]$
   $g^{-1}([7,12]) = [4,5]$

   (c) The range of h is $\{3,4,5,6,7,8,9,\dots\} = \{n \in \mathbb{N} \mid n \geq 3\}$
   $h(\{1,2,3\}) = \{3,4,5\}$
   $h^{-1}(\{1,3,5,7\}) = \{1,3,5\}$

2. The function $f \circ g$ is:

$$(f \circ g)(x) = \begin{cases} 9x^2, & \text{if} \quad x \geq 0 \\ x^2, & \text{if} \quad x < 0 \end{cases}$$

The function $g \circ f$ is:

$$(g \circ f)(x) = \begin{cases} 3x^2, & \text{if} \quad x \geq 0 \\ 3x + 15, & \text{if} \quad -5 \leq x < 0 \\ -x - 5 & \text{if} \quad x < -5 \end{cases}$$

3. (a) **Theorem.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2x + 5$. The function f is bijective.*

   Proof. First, we will prove that f is injective. Let $x, y \in \mathbb{R}$. Suppose that $f(x) = f(y)$. Then $2x + 5 = 2y + 5$. Thus, $2x = 2y$, so $x = y$. Thus, f is injective.
   Now, we will prove that f is surjective. Let $y \in \mathbb{R}$. Let $x = \frac{y-5}{2}$. Then $f(x) = f\left(\frac{y-5}{2}\right) = 2\left(\frac{y-5}{2}\right) + 5 = y - 5 + 5 = y$. Thus, f is surjective. Since f is injective and surjective, f is bijective. $\square$

   (b) **Theorem.** *Let $g\colon (0,\infty) \to \mathbb{R}$ be the function defined by $g(x) = 1/x$. The function g is injective.*

   Proof. Let $x, y \in (0, \infty)$. Suppose that $g(x) = g(y)$. Then $1/x = 1/y$. Cross-multiplying, we get that $y = x$. Thus, g is injective. $\square$

The function g is not surjective, because there is no $x \in (0, \infty)$ such that $g(x) = 0$.

(c) **Theorem.** *Let $h\colon \mathbb{R} \to [1, \infty)$ be the function defined by $h(x) = x^2 + 1$. The function h is surjective.*

Proof. Let $y \in [1, \infty)$. Let $x = \sqrt{y-1}$. (Since $y \geq 1$, we have that $y - 1 \geq 0$, so x exists.) Then, $h(x) = h\left(\sqrt{y-1}\right) = \left(\sqrt{y-1}\right)^2 + 1 = (y-1) + 1 = y$. Thus, h is surjective. $\square$

The function h is not injective, because $h(-1) = 2$ and $h(1) = 2$.

4. (a) $x = [6]$

   (b) $x = [4]$

   (c) no solutions

   (d) $x = [2]$ or $x = [5]$ or $x = [8]$

5. **Theorem.** *Let $\sim$ be the relation on $\mathbb{R}^2 - \{(0,0)\}$ defined by $(x, y) \sim (w, z)$ if and only if there exists $k \in \mathbb{R} - \{0\}$ such that $x = kw$ and $y = kz$. The relation $\sim$ is an equivalence relation.*

Proof. First, we will show that $\sim$ is reflexive. Let $(x, y) \in \mathbb{R}^2 - \{(0,0)\}$. Then, $x = 1 \cdot x$ and $y = 1 \cdot y$, so $(x, y) \sim (x, y)$. Thus, $\sim$ is reflexive.

Next, we will show that $\sim$ is symmetric. Let $(x, y), (w, z) \in \mathbb{R}^2 - \{(0,0)\}$. Suppose that $(x, y) \sim (w, z)$. Then, there exists $k \in \mathbb{R} - \{0\}$ such that $x = kw$ and $y = kz$. Then, $w = (1/k)x$ and $z = (1/k)y$. Since $1/k \in \mathbb{R} - \{0\}$, we have that $(w, z) \sim (x, y)$. Thus, $\sim$ is symmetric.

Finally, we will show that $\sim$ is transitive. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2 - \{(0,0)\}$. Suppose that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. Then, there exists $k, j \in \mathbb{R} - \{0\}$ so that $x_1 = kx_2$, $y_1 = ky_2$, $x_2 = jx_3$, and $y_2 = jy_3$. Substituting, $x_2 = jx_3$ into the equation $x_1 = kx_2$, we get that $x_1 = k(jx_3)$. Thus, $x_1 = (kj)x_3$. Substituting $y_2 = jy_3$ into the equation $y_1 = ky_2$, we get that $y_1 = k(jy_3)$. Thus, $y_1 = (kj)y_3$. Since $kj \in \mathbb{R} - \{0\}$, we have that $(x_1, y_1) \sim (x_3, y_3)$. Thus, $\sim$ is transitive. Since $\sim$ is reflexive, symmetric, and transitive, $\sim$ is an equivalence relation. $\square$

6. (a) **Theorem.** *Let $\sim$ be the relation on $\mathbb{N}$ defined by $a \sim b$ if and only if there exists $n \in \mathbb{Z}$ such that $a = 2^n b$, for all $a, b \in \mathbb{N}$. The relation $\sim$ is an equivalence relation.*

   Proof. First, we will show that $\sim$ is reflexive. Let $a \in \mathbb{N}$. Then, $a = 2^0 a$. Thus, $a \sim a$.

   Next, we will show that $\sim$ is symmetric. Let $a, b \in \mathbb{N}$. Suppose that $a \sim b$. Then, there exits $n \in \mathbb{Z}$ such that $a = 2^n b$. Then, $b = 2^{-n} a$. Since $-n$ is an integer, we have that $b \sim a$. Thus, $\sim$ is symmetric.

   Finally, we will show that $\sim$ is transitive. Let $a, b, c \in \mathbb{N}$. Suppose that $a \sim b$ and $b \sim c$. Then there exists $n, m \in \mathbb{Z}$ such that $a = 2^n b$ and $b = 2^m c$. Then, by substitution, $a = 2^n 2^m c$. Thus, $a = 2^{n+m} c$. Since $n + m$ is an integer, $a \sim c$. Thus, $\sim$ is transitive. Since $\sim$ is reflexive, symmetric, and transitive, $\sim$ is an equivalence relation. $\square$

   (b) $[0] = \{0\}$

   $$[3] = \left\{\ldots, \frac{3}{2^3}, \frac{3}{2^2}, \frac{3}{2}, 3, 3 \cdot 2, 3 \cdot 2^2, 3 \cdot 2^3, \ldots\right\}$$

7. (a) **Theorem.** *Let $n \in \mathbb{N}$. Then,* $2 + 5 + 8 + 11 + 14 + \cdots + (3n - 1) = \dfrac{(n)(3n+1)}{2}$.

   Proof. We proceed by induction on n.

   Base Case If $n = 1$, then both sides of the equation are equal to 2, so the formula holds in this case.

Induction Step Now suppose that the formula holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
2+5+8+\cdots+(3n-1)+(3(n+1)-1) &= \frac{n(3n+1)}{2}+(3(n+1)-1) \\
&= \frac{n(3n+1)}{2}+3n+2 \\
&= \frac{n(3n+1)+2(3n+2)\,]}{2} \\
&= \frac{3n^2+n+6n+4}{2} \\
&= \frac{3n^2+7n+4}{2} \\
&= \frac{(n+1)(3n+4)}{2} \\
&= \frac{(n+1)\left(3(n+1)+1\right)}{2}
\end{aligned}$$

so the formula holds for $n+1$ as well.

Therefore, by induction, the formula holds for all $n \in \mathbb{N}$. □

(b) **Theorem.** *Let* $n \in \mathbb{N}$. *Then,* $2+4+8+16+\cdots+2^n = 2^{n+1}-2$.

Proof. We proceed by induction on n.

Base Case If $n = 1$, then both sides of the equation are equal to 2, so the formula holds in this case.

Induction Step Now suppose that the formula holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
2+4+8+16+\cdots+2^n+2^{n+1} &= 2^{n+1}-2+2^{n+1} \\
&= 2\cdot 2^{n+1}-2 \\
&= 2^{n+2}-2
\end{aligned}$$

so the formula holds for $n+1$ as well.

Therefore, by induction, the formula holds for all $n \in \mathbb{N}$. □