Answers to Practice Problems

Practice Problems from Textbook:

Chapter 1:

1.2.7.
(1) True
(2) False
(3) True
(4) True
(5) True
(6) True

1.2.8.
(1) True
(2) True
(3) False
(4) True
(5) True
(6) True

1.4.1. (1)

(1) $P \wedge Q$	
(2) $(P \vee Q) \to R$	
(3) P	(1), Simplification
(4) $P \vee Q$	(3), Addition
(5) R	(2), (4), Modus Ponens

(2) This argument is invalid. If X is true, Y is true, and Z is false, then both of the premises are true, but the conclusion is false.

1.5.7. The negation is:

For all integers Q there exists a real number $x > 0$ such that for all positive integers k either $\ln(Q - x) \leq 5$ or $x \leq k$ and Q is not cacophonous.

Chapter 2:

2.2.2. (1) **Theorem.** *Let n be an integer. Then, $1|n$.*

Proof. Since $1 \cdot n = n$, we have that $1|n$. $\square$

(2) **Theorem.** *Let n be an integer. Then, $n|n$.*

Proof. Since $n \cdot 1 = n$, we have hat $n|n$. $\square$

(3) **Theorem.** *Let m and n be integers. If $m|n$, then $m|(-n)$.*

Proof. Suppose that $m|n$. Then, there exists an integer k such that $mk = n$. Multiplying this equation by -1 we have that $-mk = -n$, which is equivalent to $m(-k) = -n$. Since $-k$ is an integer, we have that $m|(-n)$. $\square$

2.2.3. (1) **Theorem.** *Let n be an integer. If n is even, then $3n$ is even.*

Proof. Suppose that n is even. Then, there exists an integer k such that $n = 2k$. Then, $3n = 3(2k) = 2(3k)$. Since $3k$ is an integer, it follows that $3n$ is even. $\square$

(2) **Theorem.** *Let n be an integer. If n is odd, then $3n$ is odd.*

Proof. Suppose that n is odd. Then, there exists an integer k such that $n = 2k+1$. Then, $3n = 3(2k + 1) = 6k + 3 = 2(3k + 1) + 1$. Since $3k + 1$ is an integer, we have that $3n$ is odd. $\square$

2.2.5. (1) **Theorem.** *Let n and m be integers. Suppose that n and m are divisible by 3. Then, $n + m$ is divisible by 3.*

Proof. Since n and m are divisible by 3, there exist integers p and q so that $n = 3p$ and $m = 3q$. Then, $n + m = 3p + 3q = 3(p + q)$. Since $p + q$ is an integer, it follows that $3|(n + m)$. $\square$

(2) **Theorem.** *Let n and m be integers. Suppose that n and m are divisible by 3. Then, nm is divisible by 3.*

Proof. Since n and m are divisible by 3, there exist integers p and q so that $n = 3p$ and $m = 3q$. Then, $nm = (3p)(3q) = 3(3pq)$. Since $3pq$ is an integer, we have that $3|nm$. $\square$

2.4.4. **Theorem.** *Let n be an integer. One of the two numbers n and $n+1$ is even, and the other is odd.*

Proof. There are two cases: either n is even or n is odd.

Case 1: Suppose that n is even. Then there exists an integer k such that $n = 2k$. Then, $n+1 = 2k+1$, so $n+1$ is odd. Thus, n is even and $n+1$ is odd.

Case 2: Suppose that n is odd. Then there exists an integer k such that $n = 2k+1$. Then, $n+1 = (2k+1)+1 = 2(k+1)$, so $n+1$ is even. Thus, $n+1$ is even and n is odd. $\square$

2.5.5. (1) **Theorem.** *For each real number x, there exists a real number y such that $e^x - y > 0$.*

Proof. Let x be an arbitrary real number. Let $y = -1$. Then, $e^x > y$, since e^x is always positive. Thus, $e^x - y > 0$. $\square$

(2) **Theorem.** *There exists a real number y such that for all real numbers x, the inequality $e^x - y > 0$ holds.*

Proof. Let $y = -1$, and let x be an arbitrary real number. Then $e^x > y$, since e^x is always positive. Thus, $e^x - y > 0$. $\square$

(3) **Theorem.** *For each real number y, there exists a real number x such that $e^x - y > 0$.*

Proof. Let y be an arbitrary real number. If $y < 1$, let $x = 0$. Then, $e^x - y = 1 - y$. Since $y < 1$, we have that $1 - y > 0$, so that $e^x - y > 0$.
If $y \geq 1$, let $x = \ln(2y)$. Then, $e^x = e^{\ln(2y)} = 2y$. Thus, $e^x - y = 2y - y = y$, which is greater than 0. $\square$

(4) The statement is not true, so we prove the negation:

Theorem. *For all real numbers x there exists a real number y such that the inequality $e^x - y \leq 0$ holds.*

Proof. Let x be an arbitrary real number. Let $y = e^x + 1$. Then:

$$e^x - y = e^x - (e^x + 1) = -1$$

Thus, $e^x - y \leq 0$. $\square$

Chapter 3:

3.2.2.
(1) False
(2) True
(3) True
(4) True
(5) False
(6) False
(7) False
(8) True
(9) True

3.2.8. We have the following subset relationships:

- $C \subseteq D$ and $C \subseteq B$
- $E \subseteq D$
- $P \subseteq D$ and $P \subseteq B$
- $N \subseteq C$, $N \subseteq D$, and $N \subseteq B$
- $S \subseteq E$ and $S \subseteq D$
- D is not a subset of any of the other sets
- $B \subseteq D$

3.3.1.
(1) $A \cup B = \{1, 2, 3, 4, 5, 7\}$
(2) $A \cap B = \{1, 3\}$
(3) $A \times B = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (5,1), (5,2), (5,3), (5,4), (7,1), (7,2), (7,3), (7,4)\}$
(4) $A - B = \{5, 7\}$
(5) $B - A = \{2, 4\}$

3.3.3.
(1) $Y \cup Z = (1, 4]$
(2) $Z \cap W = \emptyset$
(3) $Y - W = [2, 3]$
(4) $X \times W = \{(x, w) \mid 0 \leq x < 5 \text{ and } 3 < w < 5\}$
(5) $(X \cap Y) \cup Z = (1, 4]$
(6) $X - (Z \cup W) = [0, 1]$

3.3.9. **Theorem.** *Let A and B be sets. Then,* $(A \cup B) - A = B - (A \cap B)$

Proof. Let $x \in (A \cup B) - A$. Then, $x \in A \cup B$ and $x \notin A$. Since $x \in A \cup B$, we have that $x \in A$ or $x \in B$. Since we already know that $x \notin A$, we must have $x \in B$. Also, since $x \notin A$, we have that $x \notin A \cap B$. Thus, $x \in B - (A \cap B)$. Therefore, $(A \cup B) - A \subseteq B - (A \cap B)$.

Now, suppose that $x \in B - (A \cap B)$. Then, $x \in B$ and $x \notin A \cap B$. Since it is not the case that x is in both A and B, it must be the case that $x \notin A$ or $x \notin B$. Since we already know that $x \in B$, we can conclude that $X \notin A$. Also, since $x \in B$, we have that $x \in A \cup B$. Thus, we have that $x \in A \cup B$ and $x \notin A$, so we can conclude that $x \in (A \cup B) - A$. Therefore $B - (A \cap B) \subseteq (A \cup B) - A$. □

Additional Problems:

1. (1) $A \to (B \vee C)$
   (2) $\neg B$
   (3) $\neg C$

(4) $\neg B \wedge \neg C$	(2), (3), Adjunction
(5) $\neg(B \vee C)$	(4), De Morgan's Law
(5) $\neg A$	(1), (5), Modus Tollens

2. This argument is invalid. If L is false, N is true, and P is false, then all of the premises are true, but the conclusion is false.

3. (a) **Theorem.** *Let n and m be integers. If $2|n$ and $3|m$, then $6|(3n + 2m)$.*

   Proof. Suppose that $2|n$ and $3|m$. Then, there exist integers p and q such that $2p = n$ and $3q = m$. Then, $3n + 2m = 3(2p) + 2(3q) = 6p + 6q = 6(p + q)$. Since $p + q$ is an integer, we have that $6|(3n + 2m)$. □

   (b) **Theorem.** *Let n and m be integers. If $n|m$, then $n^2|m^2$.*

   Proof. Suppose that $n|m$. Then, there exists an integer p such that $np = m$. Squaring both sides of this equation, we get that $(np)^2 = m^2$, which is equivalent to $n^2(p^2) = m^2$. Since p^2 is an integer, we can conclude that $n^2|m^2$. □

(c) **Theorem.** *Let n be an integer. Then, $n^2 - n$ is even.*

Proof. There are two cases: either n is even or n is odd.

Case 1: Suppose that n is even. Then there exists an integer k so that $n = 2k$. Then, $n^2 - n = (2k)^2 - (2k) = 4k^2 - 2k = 2(2k^2 - k)$. Since $2k^2 - k$ is an integer, we have that $n^2 - n$ is even.

Case 2: Suppose that n is odd. Then there exists an integer k so that $n = 2k+1$. Then, $n^2 - n = (2k+1)^2 - (2k+1) = 4k^2 + 2k = 2(2k^2 + k)$. Since $2k^2 + k$ is an integer, we have that $n^2 - n$ is even. □

(d) **Theorem.** *Let n be an integer. If 6 does not divide $2n$, then 3 does not divide n.*

Proof. We will prove the contrapositive: if $3|n$ then $6|2n$. Suppose that $3|n$. Then there exists an integer k so that $3k = n$. Multiplying by 2, we have $6k = 2n$. Thus, $6|2n$. □

4. **Theorem.** *Let x be a non-zero rational number and let y be an irrational number. Then, $\frac{x}{y}$ is irrational.*

Proof. Proof by contradiction. Suppose that $\frac{x}{y}$ is rational. Then, there exist integers a and b so that $\frac{x}{y} = \frac{a}{b}$. Note that if $a = 0$, then $x = 0$. Since x is non-zero, we know that $a \neq 0$. Also, since x is rational, there exist integers m and n so that $x = \frac{m}{n}$. By substitution into the equation $\frac{x}{y} = \frac{a}{b}$, we have that:

$$\frac{m/n}{y} = \frac{a}{b}$$

Since $a \neq 0$, we can solve this equation for y, obtaining:

$$y = \frac{mb}{na}$$

Since mb and na are integers, we see that y is rational, contradicting our assumption that y is irrational. Thus, $\frac{x}{y}$ is irrational. □

5. (a) $\{6, 8, 9\}$

   (b) $\{1, 2, 3, 8\}$

   (c) $\{\emptyset, \{6\}, \{7\}, \{6, 7\}\}$

   (d) $\{(1,7), (1,8), (2,7), (2,8), (8,7), (8,8)\}$

6. (a) **Theorem.** *Let* A, B, C *be sets. Then,* $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

   Proof. Let $x \in (A \cup B) \cap C$. Then $x \in A \cup B$ and $x \in C$. Since $x \in A \cup B$, we have that $x \in A$ or $x \in B$.

   Case 1: Suppose that $x \in A$. Then, $x \in A \cup (B \cap C)$.

   Case 2: Suppose that $x \in B$. Since we also know that $x \in C$, we have that $x \in B \cap C$. Thus, $x \in A \cup (B \cap C)$.

   In both cases, we have that $x \in A \cup (B \cap C)$. Thus, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$. □

   (b) The statement is not true, so we will provide a counter example. Consider the following sets:

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{2\} \\ C &= \{3\} \end{aligned}$$

   Then, $A - (B \cap C) = \{1, 2, 3\}$ and $(A - B) \cap (A - C) = \{1\}$.

7. **Theorem.** *Let* A, B, C, D *be sets. Then,* $(A - B) \cup (C - D) \subseteq (A \cup C) - (B \cap D)$.

   Proof. Let $x \in (A - B) \cup (C - D)$. Then $x \in A - B$ or $x \in C - D$, so we have two cases.

   Case 1: Suppose that $x \in A - B$. Then $x \in A$ and $x \notin B$. Since $x \in A$, we have that $x \in A \cup C$. Since $x \notin B$, it is not the case that $x \in B$ and $x \in D$. Thus, $x \notin B \cap D$. Thus, $x \in (A \cup C) - (B \cap D)$.

   Case 2: Suppose that $x \in C - D$. Then $x \in C$ and $x \notin D$. Since $x \in C$, we have that $x \in A \cup C$. Since $x \notin D$, we have that $x \notin B \cap D$. Thus, $x \in (A \cup C) - (B \cap D)$.

   In both cases, we have that $x \in (A \cup C) - (B \cap D)$. Therefore, we have that $(A - B) \cup (C - D) \subseteq (A \cup C) - (B \cap D)$. □