Takehome Midterm Solutions

Math 261, Spring 2013

Problem 1

(a) The square-free integers between 1 and 25 are: 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23.

(b)(i)

Theorem. *There exists an integer a with the following properties: a is not square-free, and there does not exist an integer k such that $a = k^2$.*

Proof. Let $a = 12$. Then a is not square-free since $4|a$, and a is also not a perfect square. □

(b)(ii)

Theorem. *Let a and b be integers, and suppose that b is square-free and $a|b$. Then a is square-free as well.*

Proof. Let n be an integer, and suppose that $n^2|a$. Then there exists an integer j so that $a = jn^2$. Also, since $a|b$, there exists an integer k so that $b = ka$. Then $b = k(jn^2) = (kj)n^2$, which proves that $n^2|b$. Since b is square-free, it follows that $n = \pm 1$. □

(b)(iii)

Theorem. *Let a and b be integers, and suppose that ab is square-free. Then a and b are relatively prime.*

Proof. Let n be an integer, and suppose that $n|a$ and $n|b$. Then there exist integers j and k so that $a = nj$ and $b = nk$. Then $ab = (nj)(nk) = n^2jk$, and thus $n^2 \mid ab$. Since ab is square-free, it follows that $n = \pm 1$. □

(b)(iv)

Theorem. *For every integer a, there exists an integer b so that $a < b$ and b is not square-free.*

Proof. Let a be an integer. There are two cases: either $a \leq 1$ or $a > 1$:

- If $a \leq 1$, let $b = 4$. Then $b > a$ and b is not square-free.
- If $a > 1$, let $b = a^2$. Then clearly $a < b$. Furthermore, since $a^2|b$ and $a \neq \pm 1$, the integer b is not square-free. □

(b)(v)

Theorem. *There exists an integer k so that $k^2 + 1$ is not square-free.*

Proof. Let $k = 7$. Then $k^2 + 1 = 50$, which is divisible by 25. $\square$

Problem 2

Theorem. *Let A, B, and C be sets, and suppose that $A \subseteq B$ and $B \subseteq C$. Then*

$$C - A = (C - B) \cup (B - A).$$

Proof. Let $x \in C - A$. There are two cases: either $x \in B$ or $x \notin B$. If $x \in B$, then since $x \in C - A$ we know $x \notin A$, and therefore $x \in B - A$. If $x \notin B$, then since $x \in C - A$ we know that $x \in C$, and therefore $x \in C - B$. In either case, it follows that $x \in (C - B) \cup (B - A)$. Thus, $C - A \subseteq (C - B) \cup (B - A)$.

Now let $x \in (C - B) \cup (B - A)$. Again there are two cases: either $x \in C - B$ or $x \in B - A$. If $x \in C - B$, then we know that $x \in C$ and $x \notin B$. Since $A \subseteq B$ and $x \notin B$, it follows that $x \notin A$, and therefore $x \in C - A$. In the second case, if $x \in B - A$, then we know that $x \in B$ and $x \notin A$. Since $B \subseteq C$ and $x \in B$, it follows that $x \in C$, and therefore $x \in C - A$ in this case as well. Thus, $(C - B) \cup (B - A) \subseteq C - A$.

Therefore, $C - A = (C - B) \cup (B - A)$. $\square$

Problem 3

Let A and B be sets, and let $f\colon A \to B$ be a function.

(a)

Theorem. *If $P, Q \subseteq A$, then $f(P) - f(Q) \subseteq f(P - Q)$.*

Proof. Let $b \in f(P) - f(Q)$. Then $b \in f(P)$ and $b \notin f(Q)$. Since $b \in f(P)$, we know that $b = f(a)$ for some $a \in P$. Since $b \notin f(Q)$, we also know that $a \notin Q$, and hence $a \in P - Q$. Since $b = f(a)$, we conclude that $b \in f(P - Q)$. Therefore, $f(P) - f(Q) \subseteq f(P - Q)$. $\square$

(b)

Theorem. *If $C, D \subseteq B$, then $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.*

Proof. Let $a \in f^{-1}(C - D)$. Then $f(a) \in C - D$, and hence $f(a) \in C$ and $f(a) \notin D$. It follows that $a \in f^{-1}(C)$ and $a \notin f^{-1}(D)$, and therefore $a \in f^{-1}(C) - f^{-1}(D)$. Thus, $f^{-1}(C - D) \subseteq f^{-1}(C) - f^{-1}(D)$.

For the other direction, suppose that $a \in f^{-1}(C) - f^{-1}(D)$. Then $a \in f^{-1}(C)$ and $a \notin f^{-1}(D)$. Since $a \in f^{-1}(C)$, we know that $f(a) \in C$. Since $a \notin f^{-1}(D)$, we know that $f(a) \notin D$. Then $f(a) \in C - D$, and therefore $a \in f^{-1}(C - D)$. Thus, we have that $f^{-1}(C) - f^{-1}(D) \subseteq f^{-1}(C - D)$.

Therefore $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$. $\square$