Math 315: Answers to Practice Problems

1. (a) Suppose Player 1 plays the mixed strategy $pA + (1-p)B$. If Player 2 plays C, Player 1 gets $-4 + 9p$. If Player 2 plays D, then Player 1 gets $2 - 5p$. Setting these equal, we get that $p = 3/7$. Plugging $p = 3/7$ back into $-4+9p$, we get that the maximin $\underline{v} = -4 + 9(3/7) = -1/7$. We know that the maximin and minimax are equal for zero-sum games, so we have:

$$\underline{v} = \overline{v} = -\frac{1}{7}$$

   (b) As found in part (a), Player 1's optimal strategy is $\frac{3}{7}A + \frac{4}{7}B$.
   Now, we find Player 2's optimal strategy. Suppose that Player 2 plays the mixed strategy $pC + (1-p)D$. If Player 1 plays A, then Player 1 gets $-3+8p$. If Player 1 plays B, then Player 2 gets $2 - 6p$. Setting these equal, we get that $p = 5/14$. Thus, Player 2's optimal strategy is $\frac{5}{14}C + \frac{9}{14}D$.

   (c) If both players play their optimal strategy, Player 1's expected payoff is $-1/7$ (the maximin and minimax from part (a)) and Player 2's expected payoff is $1/7$.

   (d) The Nash equilibrium for this game is $\left(\frac{3}{7}A + \frac{4}{7}B, \frac{5}{14}C + \frac{9}{14}D\right)$. (In a zero-sum game, the optimal strategies form the Nash equilibria.)

2. (a) Suppose Player 1 plays the mixed strategy $pA + (1-p)B$. If Player 2 plays C, Player 1 gets $4p$. If player 2 plays D, Player 1 gets $3 - 2p$. When we graph these lines, we see that the maximin occurs at the intersection of the two lines. Setting them equal, we get that $p = 1/2$. We can plug $p = 1/2$ back into $4p$, and we get that the maximin $\underline{v}_1 = 2$. In a 2-player game, the maximin and minimax for a given player will be equal. Thus, we have:

$$\underline{v}_1 = \overline{v}_1 = 2$$

   The strategy that guarantees Player 1 at least $\underline{v}_1$ is $\frac{1}{2}A + \frac{1}{2}B$.

   (b) Suppose Player 2 plays the mixed strategy $pC + (1-p)D$. If Player 1 plays A, Player 2 gets $5 - 2p$. If Player 1 plays B, Player 2 gets $6 - 4p$. When we graph these, we see that the maximin does not occur at the intersection point, but instead occurs at $p = 0$ with $\underline{v}_2 = 5$. In a 2-player game, the maximin and minimax for a given player will be equal. Thus, we have :

$$\underline{v}_2 = \overline{v}_2 = 5$$

   The strategy that guarantees Player 2 at least $\underline{v}_2$ is D (this is the strategy when $p = 0$).

3. (a) The pure strategy maximin and minimax for Player 1 are $\underline{v}_1 = 0$ and $\overline{v}_1 = 2$.
   (b) The pure strategy maximin and minimax for Player 2 are $\underline{v}_2 = 1$ and $\overline{v}_2 = 6$.
   (c) To find Player 1's mixed strategy maximin $\underline{v}_1$, we would solve the following linear program:

$$\begin{array}{l} \text{Max} \quad z \\ \text{subject to} \\ -3x_1 + 2x_3 \geq z \\ 5x_1 + 4x_2 - x_3 \geq z \\ -2x_1 + 5x_2 + 3x_3 \geq z \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{array}$$

   (d) To find Player 2's mixed strategy maximin $\underline{v}_2$, we would solve the following linear program:

$$\begin{array}{l} \text{Max} \quad z \\ \text{subject to} \\ 11x_1 + 4x_2 + 9x_3 \geq z \\ 5x_1 - 3x_2 + 6x_3 \geq z \\ x_1 + 7x_2 - x_3 \geq z \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{array}$$

4. (a) The pure strategy maximin are:

$$\underline{v}_1 = 2$$
$$\underline{v}_2 = 1$$
$$\underline{v}_3 = 3$$

   (b) The pure strategy minimax are:

$$\overline{v}_1 = 3$$
$$\overline{v}_2 = 4$$
$$\overline{v}_3 = 3$$

5. (a) This game is not zero-sum. The linear function $f(x) = -3x + 1$ would change the first three payoffs for Player 1 into the negative of the first three payoffs for Player 2, but it would not work for the payoff where both players choose B. In addition, the function $f(x) = -3x + 1$ is not a positive affine transformation, since -3 is not positive.

   (b) This game is zero-sum. The positive affine transformation $f(x) = 2x - 5$ will transform the payoffs for Player 1 into the negative of the payoffs for Player 2.

6. (a) No, there is not such a Nash equilibrium.

   Suppose Player 1 plays a mixed strategy of the form $pA + (1 - p)B$. If Player 2 plays A, then Player 2's expected payoff is $-10 + 20p$. If Player 2 plays B, then Player 2's expected payoff is $-4 + 4p$. Setting these equal, we get $p = 3/8$.

   Suppose Player 2 plays a mixed strategy of the form $pA + (1 - p)B$. If Player 1 plays A, then Player 1's expected payoff is $8 - 9p$. If Player 1 plays B, then Player 1's expected payoff is $2 - 2p$. Setting these equal, we get $p = 6/7$.

   We now check whether these strategies give a Nash equilibrium. Suppose Player 1 uses the strategy $\frac{3}{8}A + \frac{5}{8}B$. Then Player 2's payoffs for playing A, B, and C are:

$$\begin{aligned} A: &\quad \tfrac{3}{8}(10) + \tfrac{5}{8}(-10) = \tfrac{-5}{2} \\ B: &\quad \tfrac{3}{8}(0) + \tfrac{5}{8}(-4) = \tfrac{-5}{2} \\ C: &\quad \tfrac{3}{8}(-2) + \tfrac{5}{8}(8) = \tfrac{17}{4} \end{aligned}$$

   Since $\frac{17}{4} > \frac{-5}{2}$, Player 2 would prefer to play C over a mixed strategy involving A and B, so these strategies do not give a Nash equilibrium, and there is not a mixed strategy Nash equilibrium where both players use A and B with non-zero probability and do not use C.

   (b) Suppose that Player 1 uses the mixed strategy $x_1A + x_2B + x_3C$. If Player 2 uses strategy A, then Player 2's expected payoff is $10x_1 - 10x_2 + 2x_3$. If Player 2 uses strategy B, then Player 2's expected payoff is $-4x_2 + 4x_3$. If Player 2 uses strategy C, then Player 2's expected payoff is $-2x_1 + 8x_2 - 10x_3$. In order for this to be part of a Nash equilibrium in which Player 2 uses strategies A and C with non-zero probability and does not use strategy B, we need the following equations and inequalities to be satisfied:

$$\begin{aligned} &10x_1 - 10x_2 + 2x_3 = -2x_1 + 8x_2 - 10x_3 \\ &10x_1 - 10x_2 + 2x_3 \geq -4x_2 + 4x_3 \\ &x_1 + x_2 + x_3 = 1 \\ &0 \leq x_1, x_2, x_3 \leq 1 \end{aligned}$$

These simplify to:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 0 \\ 10x_1 - 6x_2 - 2x_3 &\geq 0 \\ x_1 + x_2 + x_3 &= 1 \\ 0 \leq x_1, x_2, x_3 &\leq 1 \end{aligned}$$

If we solve the two equations above ($2x_1 - 3x_2 + 2x_3 = 0$ and $x_1 + x_2 + x_3 = 1$) we get the following solutions (with t as a parameter): $x_1 = \frac{3}{5} - t$, $x_2 = \frac{2}{5}$, $x_3 = t$. Since $x_3 = t$, we know that $0 \leq t \leq 1$. We can plug these into the above inequality:

$$10\left(\tfrac{3}{5} - t\right) - 6\left(\tfrac{2}{5}\right) - 2t \geq 0$$

Solving, we get that $t \leq 3/10$. Thus, we want $0 \leq t \leq 3/10$.

Now, we suppose that Player 2 uses a mixed strategy $y_1A + y_2C$. Then, if Player 1 plays A, Player 1's expected payoff is $-y_1 + 4y_2$. If Player 2 plays B, Player 1's expected payoff is $3y_2$. If Player 1 plays C, Player 1's expected payoff is $-2y_1+5y_2$. In order for this to be a Nash equilibrium where Player 1 uses a mixed strategy involving A, B, and C, we need all three of these to be equal. Thus, we have the following system of equations:

$$\begin{aligned} -y_1 + 4y_2 &= 3y_2 \\ -y_1 + 4y_2 &= -2y_1 + 5y_2 \\ y_1 + y_2 &= 1 \end{aligned}$$

Solving, we get that $y_1 = 1/2$ and $y_2 = 1/2$.

Thus, we have infinitely many Nash equilibira, in which Player 1 uses the strategy $(\frac{3}{5} - t)A + \frac{2}{5}B + tC$ where $0 < t \leq \frac{3}{10}$ and where Player 2 uses the strategy $\frac{1}{2}A + \frac{1}{2}C$. The problem just asked for one such Nash equilibrium so any of these Nash equilibria would be correct.

7. (a) There is one Nash equilibria: (A, A). (Note that row A strictly dominates row B, and after eliminating row B, column A strictly dominates column B, leaving only (A, A). This shows that this is the only Nash equilibrium.)

   The strategy A is an Evolutionarily Stable Strategy, as $u_1(x, A) < u_1(A, A)$ for all mixed strategies x where $x \neq A$.

   (b) There are three Nash equilibria: (A, A), (B, B), and $\left(\frac{2}{5}A + \frac{3}{5}B, \frac{2}{5}A + \frac{3}{5}B\right)$.

   The strategies A and B are Evolutionarily Stable Strategies. The strategy $\frac{2}{5}A+\frac{3}{5}B$ is not an Evolutionarily Stable Strategy.

The strategies A and B are evolutionarily stable, because $u_1(x, A) < u_1(A, A)$ for all mixed strategies $x \neq A$ and $u_1(x, B) < u_1(B, B)$ for all mixed strategies $x \neq B$.

To see that $\frac{2}{5}A + \frac{3}{5}B$ is not evolutionarily stable, we let x be a mixed strategy $pA + (1-p)B$, and we let $x^* = \frac{2}{5}A + \frac{3}{5}B$. We note that $u_1(x, x^*) = u_1(x^*, x^*)$. Thus, we need to determine whether $u_1(x, x) < u_1(x^*, x)$ for all strategies $x \neq x^*$. We can compute $u_1(x, x)$ and $u_1(x^*, x)$:

$$\begin{aligned} u_1(x, x) &= 4p^2 + p(1-p) + 2(1-p)^2 \\ &= 5p^2 - 3p + 2 \end{aligned}$$

$$\begin{aligned} u_1(x^*, x) &= \tfrac{8}{5}p + \tfrac{3}{5}p + \tfrac{6}{5}(1-p) \\ &= \tfrac{6}{5} + p \end{aligned}$$

Thus, the inequality $u_1(x, x) < u_1(x^*, x)$ becomes:

$$5p^2 - 3p + 2 < \tfrac{6}{5} + p$$

This is equivalent to:

$$5p^2 - 4p + \tfrac{4}{5} < 0$$

We see that this inequality does not hold when $p = 0$. Thus, $\frac{2}{5}A + \frac{3}{5}B$ is not an Evolutionarily Stable Strategy.

(c) The Nash equilibria are (A, B), (B, A), and $\left(\frac{2}{3}A + \frac{1}{3}B, \frac{2}{3}A + \frac{1}{3}B\right)$.

The strategy $\frac{2}{3}A + \frac{1}{3}B$ is an Evolutionarily Stable Strategy. (The Nash equilibria (A, B) and (B, A) do not give any Evolutionarily Stable Strategies since they are not symmetric.)

To see that $\frac{2}{3}A + \frac{1}{3}B$ is evolutionarily stable, we let x be a mixed strategy $pA + (1-p)B$, and we let $x^* = \frac{2}{3}A + \frac{1}{3}B$. We note that $u_1(x, x^*) = u_1(x^*, x^*)$. Thus, we need to determine whether $u_1(x, x) < u_1(x^*, x)$ for all strategies $x \neq x^*$. We can compute $u_1(x, x)$ and $u_1(x^*, x)$:

$$\begin{aligned} u_1(x, x) &= -1p^2 + 4p(1-p) + 2(1-p)^2 \\ &= -3p^2 + 2 \end{aligned}$$

$$\begin{aligned} u_1(x^*, x) &= -\tfrac{2}{3}p + \tfrac{8}{3}(1-p) + \tfrac{2}{3} - \tfrac{2}{3}p \\ &= \tfrac{10}{3} - 4p \end{aligned}$$

Thus, the inequality $u_1(x, x) < u_1(x^*, x)$ becomes:

$$-3p^2 + 2 < \tfrac{10}{3} - 4p$$

This is equivalent to:

$$-3p^2 + 4p - \tfrac{4}{3} < 0$$

This factors as:

$$-3\left(p - \tfrac{2}{3}\right)^2 < 0$$

We see that this inequality is true for all $p \neq \frac{2}{3}$. When $p = \frac{2}{3}$, we have that $x = x^*$, so we have that $u_1(x, x) < u_1(x^*, x)$ for all strategies $x \neq x^*$. Thus, $\frac{2}{3}A + \frac{1}{3}B$ is an Evolutionarily Stable Strategy.

8. (a) If both players play A every round, then Player 1 gets 6 every round and Player 2 gets 7 every round. Player 1's expected payoff is:

$$6 + 6\lambda + 6\lambda^2 + 6\lambda^3 + \cdots = \frac{6}{1 - \lambda}$$

Player 2's expected payoff is:

$$7 + 7\lambda + 7\lambda^2 + 7\lambda^3 + \cdots = \frac{7}{1 - \lambda}$$

(b) Player 1 will get 0 for the first two rounds, and then get 3 every round after that. Player 2 will get 10 for the first two round and then get 5 every round after that. Player 1's expected payoff is:

$$3\lambda^2 + 3\lambda^3 + 3\lambda^4 + \cdots = \frac{3\lambda^2}{1 - \lambda}$$

Player 2's expected payoff is:

$$10 + 10\lambda + 5\lambda^2 + 5\lambda^3 + 5\lambda^4 + \cdots = 10 + 10\lambda + \frac{5\lambda^2}{1 - \lambda}$$

(c) If both players alternate A and B with Player 1 starting with A and Player 2 starting with B, then Player 1 will receive 0 on the first round, third round, fifth round, and so on, and will receive 10 on the second round, fourth round, sixth round and so on. Player 1's expected payoff is:

$$10\lambda + 10\lambda^3 + 10\lambda^5 + \cdots = \frac{10\lambda}{1 - \lambda^2}$$

Player 2 will receive 10 on the first round, third round, fifth round, and so on, and will receive 0 on the other rounds. Player 2's expected payoff is:

$$10 + 10\lambda^2 + 10\lambda^4 + 10\lambda^6 + \cdots = \frac{10}{1 - \lambda^2}$$

9. Suppose that both players play the Grim Strategy, then both players will cooperate each round, and both players will have payoff $\frac{-1}{1-\lambda}$. Suppose that Player 2 decides to switch to a different strategy. Any strategy in which Player 2 cooperates every round will have the same payoff as the Grim Strategy. If Player 2 defects at some point, Player 1 will also switch to defecting. At that point, there will be no advantage to switching back to Cooperate, since Player 1 will just continue to Defect, so playing Cooperate will give Player 2 a payoff of -3 for that round instead of a payoff of -2. This means that we just need to consider strategies where Player 2 plays Cooperate for n rounds (n could be zero) and then switches to Defect and plays Defect from then on.

   We will compare the payoff for this strategy (Player 2 plays Cooperate for n rounds and the plays Defect from then on) to the payoff for the Grim Strategy, and see for which values of λ, Player 2 prefers the Grim Strategy (that will tell us for what values of λ the Grim Strategy is a Nash equilibrium).

   If Player 2 plays Cooperate for n rounds, and then plays Defect, while Player 1 plays the Grim strategy, Player 2 will receive -1 for the first n rounds, and then receive 0 for the first round they Defect, and then receive -2 from then on. This makes Player 2's payoff:

$$-1-\lambda-\lambda^2-\cdots-\lambda^{n-1}+\frac{-2\lambda^{n+1}}{1-\lambda}$$

   If Player 2 plays the Grim Strategy, then we know Player 2's expected payoff is $\frac{-1}{1-\lambda}$. By thinking of the first n terms separately, we could also write the payoff for the Grim Strategy as:

$$-1-\lambda-\lambda^2-\cdots-\lambda^{n-1}-\frac{\lambda^n}{1-\lambda}$$

   We want to know for what values of λ the Grim Strategy payoff is better than the other strategy, so we want:

$$-1-\lambda-\lambda^2-\cdots-\lambda^{n-1}-\frac{\lambda^n}{1-\lambda}\geq-1-\lambda-\lambda^2-\cdots-\lambda^{n-1}+\frac{-2\lambda^{n+1}}{1-\lambda}$$

   We can cancel the first n terms:

$$-\frac{\lambda^n}{1-\lambda}\geq\frac{-2\lambda^{n+1}}{1-\lambda}$$

   Then, we can multiply both sides by $1-\lambda$ (since $\lambda<1$, this does not change the direction of the inequality):

$$-\lambda^n\geq-2\lambda^{n+1}$$

   Then, we can divide both sides by $-2\lambda^n$, which does change the direction of the inequality:

$$\frac{1}{2}\leq\lambda$$

Thus, we see that (Grim Strategy, Grim Strategy) is a Nash equilibrium for $\frac{1}{2} \leq \lambda < 1$.

10. (a) We can create this game by first adding the following games:

Coalition	A	AC	AB	ABC
Wealth	10	10	10	10

Coalition	BC	ABC
Wealth	90	90

Coalition	AC	ABC
Wealth	20	20

Coalition	AB	ABC
Wealth	50	50

And then subtracting the following game:

Coalition	ABC
Wealth	30

Then, the Shapley allocation assigns A: $10 + 10 + 25 - 10 = 35$. And assigns B: $45 + 25 - 10 = 60$. And assigns C: $45 + 10 - 10 = 45$. Thus, the Shapley allocation is:

A	35
B	60
C	45

(b) Yes, the allocation is rational. We see that A receives 35, which is more than A can get by itself. We see that B and C together receive 95, which is more than the coalition BC can get on its own. We see that A and C together receive 80, which is more than the coalition AC can get on its own. We see that A and B together receive 95, which is more than the coalition AB can get on its own.

(c) To find the Nucleolus allocation, we look at the excess for each possible coalition (each subset of $\{A, B, C\}$ not including the whole set or the null set):

Coalition	Excess
AB	35
AC	50
BC	15
A	25
B	60
C	45

The smallest excess is currently the 15 for the coalition BC. We can increase that coalition's excess by increasing the allocations for B and C while decreasing the allocation for A. We can decrease A by 5 and increase B by 2 and C by 3. The new allocation is A 30, B 62, C 48. The new excess is:

Coalition	Excess
AB	32
AC	48
BC	20
A	20
B	62
C	48

The smallest excesses are now the 20's for coalitions BC and A. We cannot improve those any, so we look at the next smallest excess which is the 32 for coalition AB. We can improve that by increasing the allocation for B and decreasing the allocation for C. We can increase B's allocation by 8, and decrease C's allocation by 8. The new allocation is A 30, B 70, C 40. The new excess is:

Coalition	Excess
AB	40
AC	40
BC	20
A	20
B	70
C	40

We cannot improve the smallest excesses any further, so this is the Nucleolus allocation:

A	30
B	70
C	40

11. **Theorem.** *A constant-sum game is strategically equivalent to a zero-sum game.*

Proof. Consider a constant-sum game, in which the sum of the payoffs equals a constant C. Let S_1 and S_2 be the set of strategy choices for Player 1 and Player 2, respectively, and let u_1 and u_2 be the payoff functions for Player 1 and Player 2, respectively. Since the game is constant-sum, for all strategy choices $s_1 \in S_1$ and $s_2 \in S_2$, we have that $u_1(s_1, s_2) + u_2(s_1, s_2) = C$. Consider the positive affine transformation $v_1 = u_1 - C$ (ie, we subtract C from each of Player 1's payoffs), and consider the game with payoff functions v_1 for Player 1 and u_2 for Player 2. This game is strategically equivalent to our original game. Also, it is a zero-sum game, because:

$$v_1(s_1, s_2) + u_2(s_1, s_2) = u_1(s_1, s_2) - C + u_2(s_1, s_2) = C - C = 0$$

Thus, every constant-sum game is strategically equivalent to a zero-sum game. □