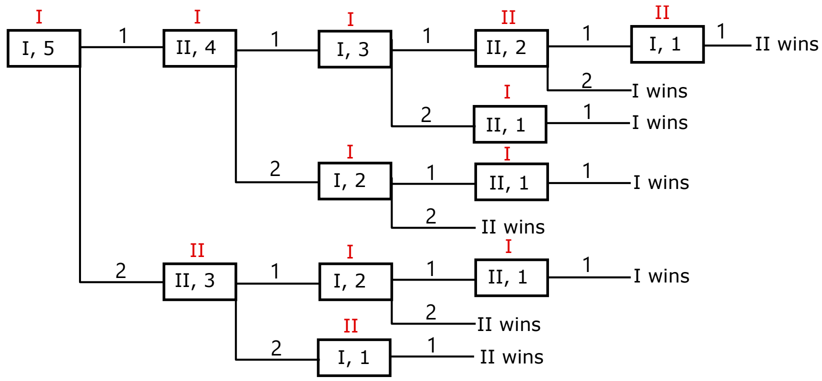
Math 315: Answers to Practice Problems

1. On the game tree, we will call the first player I and the second player II. The game tree is:



The current player (I or II) and the current number of beans remaining is listed inside each vertex. In addition, in red above each vertex is listed the player who has a winning strategy from that vertex (obtained via backward induction). (This information in red is not necessary as part of the game tree; it is just helpful in determining which player has a winning strategy.)

Player I has a winning strategy. Player I's strategy is to first remove 1 bean. Afterwards, if player II takes 1 bean, player I will take 2 beans, leaving Player II forced to take the last bean. If player II instead takes 2 beans, player I will take 1 bean, leaving player II forced to take the last bean.

2. (a) The second player has the winning strategy, because when we write the numbers in binary, there are an even number of 1's in each column:

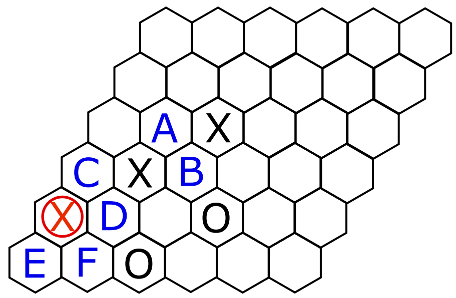
$$\begin{array}{rcl} 22 &=& 16 \quad\quad + 4 + 2 \\ 19 &=& 16 \quad\quad\quad\quad + 2 + 1 \\ 14 &=& \quad 8 + 4 + 2 \\ 11 &=& \quad 8 \quad\quad + 2 + 1 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} 22 \text{ (decimal)} &=& 10110 \text{ (binary)} \\ 19 \text{ (decimal)} &=& 10011 \text{ (binary)} \\ 14 \text{ (decimal)} &=& 01110 \text{ (binary)} \\ 11 \text{ (decimal)} &=& 01011 \text{ (binary)} \end{array}$$

(b) The second player should remove 14 beans from the pile with 22 beans. After the six beans are removed, we have heaps of size 22, 13, 14, and 11.

$$\begin{array}{rcl} 22 &=& 16 \quad\quad + 4 + 2 \\ 13 &=& \quad 8 + 4 \quad\quad + 1 \\ 14 &=& \quad 8 + 4 + 2 \\ 11 &=& \quad 8 \quad\quad + 2 + 1 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} 22 \text{ (decimal)} &=& 10110 \text{ (binary)} \\ 13 \text{ (decimal)} &=& 01101 \text{ (binary)} \\ 14 \text{ (decimal)} &=& 01110 \text{ (binary)} \\ 11 \text{ (decimal)} &=& 01011 \text{ (binary)} \end{array}$$

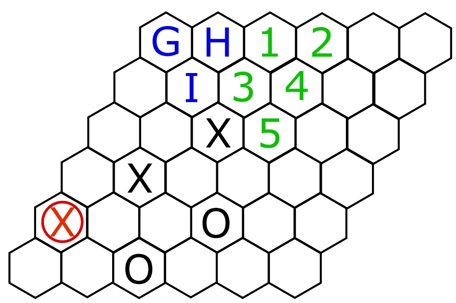
There are an odd number of 1's in the 16's place, 8's place, 4's place, and 2's place. Removing 14 beans from the pile with 22 beans will create an even number of 1's in each column.

3. Player 1 should go in the spot indicated by the circled red X:



Player 1's plan is to connect to the bottom using the X's already on the board along with either spot A or B, spot C or D, and spot E or F. If Player 2 takes one of the above spots, Player 1 can take the corresponding spot. Thus, Player 1 can ensure a path to the bottom

Player 1 can also ensure a path to the top. We will use the following labeling to describe Player 1's strategy:



If Player 2 does not take one of G, H, or I, then Player 1 can take I, and then create a path to the top using I along with either G or H.

If Player 2 takes one of G, H, or I, Player 1 can take the hexagaon labeled 4. Then, he can create a path to the top using either 3 or 5, and either 1 or 2.

4. (a) She would choose the lottery, because her utility of the lottery is:

$$u([0.5(A), 0.5(C)]) = 0.5u(A) + 0.5u(C) = 0.5(0) + 0.5(80) = 40$$

Thus, the utility of the lottery is 40, which is greater than the utility of B.

(b) First, we compute the utility of the lotteries:

$$u(L_1) = 0.3(30) + 0.7(100) = 79$$
$$u(L_2) = 0.2(0) + 0.3(30) + 0.5(80) = 49$$

Thus, Sarah prefers lottery L_1.

(c) With the compound lottery, the probability of A is $0.5(0) + 0.5(0.2) = 0.1$, the probability of B is $0.5(0.3) + 0.5(0.3) = 0.3$, the probability of C is $0.5(0) + 0.5(0.5) = 0.25$, and the probability of D is $0.5(0.7) + 0.5(0) = 0.35$. Thus Sarah is indifferent between L and the following lottery:

$$L_3 = [0.1(A), 0.3(B), 0.25(C), 0.35(D)]$$

We can also compute $u(\widehat{L})$:

$$u(\widehat{L}) = 0.5u(L_1) + 0.5u(L_2) = 0.5(79) + 0.5(49) = 64$$

(Note that it would also work to find $u(\widehat{L})$ by computing $u(L_3)$.)

(d) The cardinal utilities would be $v(A) = 10$, $v(B) = 25$, $v(C) = 50$, and $v(D) = 60$. To find these, we first write v as a positive affine tranformation of u. We find that $v = \frac{1}{2}u + 10$, which allows us to find the values for v.

5. (a) This player is risk averse.

   (b) This player is risk seeking.

6. (a) To start with, column A dominates column D. After we remove column D, row C dominates row A. After we remove row A, column C dominates column E. We are left with the following matrix:

Player 2

Player 1		A	B	C
	B	$7, 9$	$7, 3$	$1, 7$
	C	$7, 5$	$10, 10$	$3, 5$

   (b) Player 1's security level is $\underline{v}_1 = 2$. Strategy C guarantees Player 1 at least 2.

   (c) Player 2's security level is $\underline{v}_2 = 3$. Strategies C and E guarantee Player 2 at least 3.

   (d) There are two pure strategy Nash equlibiria: (B, A) and (C, B).

7. Player 1 has 4 strategies: AC, AD, BE, and BF. (It would also be correct to have 8 strategies for Player 1: ACE, ACF, ADE, ADF, BCE, BCF, BDE, and BDF.)

   Player 2 has 6 strategies: ad, ae, bd, be, cd, ce.

   The game in strategic-form is:

Player 2

Player 1		ad	ae	bd	be	cd	ce
	AC	$2,3$	$2,3$	$1,5$	$1,5$	$5,3$	$5,3$
	AD	$2,3$	$2,3$	$1,5$	$1,5$	$4,1$	$4,1$
	BE	$3,5$	$1,6$	$3,5$	$1,6$	$3,5$	$1,6$
	BF	$4,1$	$2,5$	$4,1$	$2,5$	$4,1$	$2,5$

   As mentioned above, it would also be correct to have 8 strategies for Player 1.

8. (a) Yes, it is a Nash equilibrium. To see this, we start by assuming that Player 1 uses the strategy $\frac{4}{9}$Rock + $\frac{4}{9}$Paper + $\frac{1}{9}$Scissors, and we determine Player 2's expected value for choosing Rock, Paper, or Scissors:

   - If Player 2 chooses Rock, his expected value is: $\frac{4}{9}(0) + \frac{4}{9}(-1) + \frac{1}{9}(4) = 0$
   - If Player 2 chooses Paper, his expected value is: $\frac{4}{9}(1) + \frac{4}{9}(0) + \frac{1}{9}(-4) = 0$
   - If Player 2 chooses Scissors, his expected value is: $\frac{4}{9}(-4) + \frac{4}{9}(4) + \frac{1}{9}(0) = 0$

   Player 2 is indifferent between choosing Rock, Paper, or Scissors, so the mixed strategy $\frac{4}{9}$Rock + $\frac{4}{9}$Paper + $\frac{1}{9}$Scissors also gives him the same expected value.

   Since the game is symmetric, we'll get the same results if we assume that Player 2 uses the $\frac{4}{9}$Rock + $\frac{4}{9}$Paper + $\frac{1}{9}$Scissors — Player 1 will have an expected value of 0 regardless of whether he chooses Rock, Paper, Scissors, or a mixed strategy.

   Thus, $\left(\frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}, \frac{4}{9}\text{Rock} + \frac{4}{9}\text{Paper} + \frac{1}{9}\text{Scissors}\right)$ is a Nash equilibrium.

   (b) No, it is not a Nash equilibrium. To see this, we start by assuming that Player 1 uses the strategy $\frac{5}{9}$Rock + $\frac{4}{9}$Paper, and we determine Player 2's expected value if he chooses Rock, Paper, or Scissors:

   - If Player 2 chooses Rock, Player 2's expected value is: $\frac{5}{9}(0) + \frac{4}{9}(-1) = \frac{-4}{9}$
   - If Player 2 chooses Paper, Player 2's expected value is: $\frac{5}{9}(1) + \frac{4}{9}(0) = \frac{5}{9}$
   - If Player 2 chooses Scissors, Player 2's expected value is: $\frac{5}{9}(-4) + \frac{4}{9}(4) = \frac{-4}{9}$

   Thus, Player 2's best response is Paper, so he would not use the strategy $\frac{8}{9}$Rock + $\frac{1}{9}$Scissors, so it is not a Nash equilibrium.

   (c) If Player 1 uses the strategy $\frac{5}{9}$Rock + $\frac{4}{9}$Paper, then:

   - If Player 2 chooses Rock, Player 2's expected value is: $\frac{5}{9}(0) + \frac{4}{9}(-1) = \frac{-4}{9}$

- If Player 2 chooses Paper, Player 2's expected value is: $\frac{5}{9}(1) + \frac{4}{9}(0) = \frac{5}{9}$
- If Player 2 chooses Scissors, Player 2's expected value is: $\frac{5}{9}(-4) + \frac{4}{9}(4) = \frac{-4}{9}$

Thus, if Player 2 uses the strategy $\frac{8}{9}$Rock + $\frac{1}{9}$Scissors, then Player 2's expected payoff is $\frac{8}{9}\left(\frac{-4}{9}\right) + \frac{1}{9}\left(\frac{-4}{9}\right) = \frac{-36}{81} = \frac{-4}{9}$.

If Player 2 uses the strategy $\frac{8}{9}$Rock + $\frac{1}{9}$Scissors, then:

- If Player 1 chooses Rock, Player 1's expected value is: $\frac{8}{9}(0) + \frac{1}{9}(4) = \frac{4}{9}$
- If Player 1 chooses Paper, Player 1's expected value is: $\frac{8}{9}(1) + \frac{1}{9}(-4) = \frac{4}{9}$
- If Player 1 chooses Scissors, Player 1's expected value is: $\frac{8}{9}(-4) + \frac{1}{9}(0) = \frac{-32}{9}$

Thus, if Player 1 uses the strategy $\frac{5}{9}$Rock + $\frac{4}{9}$Paper, then Player 1's expected payoff is $\frac{5}{9}\left(\frac{4}{9}\right) + \frac{4}{9}\left(\frac{4}{9}\right) = \frac{36}{81} = \frac{4}{9}$.

Thus, Player 1's expected payoff is $\frac{4}{9}$ and Player 1's expected payoff is $\frac{-4}{9}$.

9. First, we find the pure strategy Nash Equilibria. There are two pure strategy Nash equilibria: (A, D) and (B, C).

   Next, we check for mixed strategy Nash equilibria. Suppose that Player 1 uses the mixed strategy $pA + (1-p)B$. Then, Player 2's expected payoffs are:

   - If Player 2 chooses C, Player 2's expected value is: $p(6) + (1-p)(5) = 5 + p$
   - If Player 2 chooses D, Player 2's expected value is: $p(8) + (1-p)(4) = 4 + 4p$

   We set these equal:

   $$5 + p = 4 + 4p$$

   Solving, we get that $p = 1/3$. Thus, if Player 1 uses the strategy $\frac{1}{3}A + \frac{2}{3}B$, Player 2 will be indifferent between C and D (so that any mixed strategy involving C and D will be a best response for Player 2).

   Now, suppose that Player 2 uses the mixed strategy $pC + (1-p)D$. Then, Player 1's expected payoffs are:

   - If Player 1 chooses C, Player 1's expected value is: $p(1) + (1-p)(2) = 2 - p$
   - If Player 1 chooses D, Player 1's expected value is: $p(4) + (1-p)(1) = 1 + 3p$

   We set these equal:

   $$2 - p = 1 + 3p$$

   Solving, we get that $p = 1/4$. Thus, if Player 2 uses the strategy $\frac{1}{4}C + \frac{3}{4}D$, Player 1 will be indifferent between A and B (so that any mixed strategy involving A and B will be a best response for Player 1).

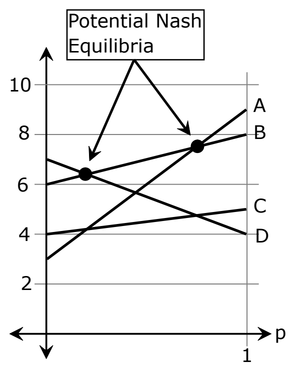
   Thus, there are three Nash equilibria: (A, D), (B, C), and $\left(\frac{1}{3}A + \frac{2}{3}B, \frac{1}{4}C + \frac{3}{4}D\right)$.

10. First, we note that there are two pure strategy Nash equilibria: (A, A) and (D, B).

Next, we look for mixed strategy Nash equilibria. Suppose that Player 2 uses the mixed strategy $pA + (1 - p)B$. Then, Player 1's expected payoffs are:

- If Player 1 chooses A, Player 1's expected value is: $p(9) + (1 - p)(3) = 3 + 6p$
- If Player 1 chooses B, Player 1's expected value is: $p(8) + (1 - p)(6) = 6 + 2p$
- If Player 1 chooses C, Player 1's expected value is: $p(5) + (1 - p)(4) = 4 + p$
- If Player 1 chooses D, Player 1's expected value is: $p(4) + (1 - p)(7) = 7 - 3p$

We graph the equations for these lines:



From the graph, we can see that there are two potential mixed strategy Nash equilibria (not counting pure strategy Nash equilibria). One potential Nash equilibrium comes from Player 1 using strategies A and B; the other comes from Player 1 using strategies B and D. We will check both of these separately.

If Player 1 uses strategies A and B, the matrix looks like:

		Player 2	
		A	B
Player 1	A	$9, 3$	$3, 1$
	B	$8, 2$	$6, 8$

Suppose that Player 1 uses the mixed strategy $pA+(1-p)B$. Then, Player 2's expected payoffs are:

- If Player 2 chooses A, Player 2's expected value is: $p(3) + (1 - p)(2) = 2 + p$
- If Player 2 chooses B, Player 2's expected value is: $p(1) + (1 - p)(8) = 8 - 7p$

We set these equal:

$$2 + p = 8 - 7p$$

Solving, we get that $p = 3/4$. Thus, if Player 1 uses the strategy $\frac{3}{4}A + \frac{1}{4}B$, Player 2 will be indifferent between A and B (so that any mixed strategy involving A and B will be a best response for Player 2).

To find the corresponding strategy for Player 2 to use, we set the equations for the lines for A and B equal:

$$3 + 6p = 6 + 2p$$

Solving, we get that $p = 3/4$. Thus, if Player 2 uses the strategy $\frac{3}{4}A + \frac{1}{4}B$, Player 1's best response will be A, B, or any combination of A and B (we can see this by looking at the previous graph). Thus, one mixed strategy Nash equilibrium is $\left(\frac{3}{4}A + \frac{1}{4}B, \frac{3}{4}A + \frac{1}{4}B\right)$.

Now, we consider Player 1 using strategies B and D. If Player 1 uses B and D, the matrix looks like:

Player 2

		A	B
Player 1	B	$8, 2$	$6, 8$
	D	$4, 5$	$7, 9$

Suppose that Player 1 uses the mixed strategy $pB + (1-p)D$. Then, Player 2's expected payoffs are:

- If Player 2 chooses A, Player 2's expected value is: $p(2) + (1-p)(5) = 5 - 3p$
- If Player 2 chooses B, Player 2's expected value is: $p(5) + (1-p)(9) = 9 - 4p$

We set these equal:

$$5 - 3p = 9 - 4p$$

Solving, we get that $p = 4$. Since $4 > 1$, this does not correspond to a Nash equilibrium. Thus, there is not a Nash equilibrium corresponding to the second point in the graph.

Thus, this game has three Nash equilibria: (A, A), (D, B), and $\left(\frac{3}{4}A + \frac{1}{4}B, \frac{3}{4}A + \frac{1}{4}B\right)$.

11. (a) The following strategic-form game is one example that has no pure strategy Nash equilibria.

Player 2

		A	B
Player 1	A	$3, 2$	$2, 3$
	B	$1, 3$	$3, 1$

(b) The following strategic-form game is one example:

Player 2

		A	B	C
Player 1	A	$2,1$	$1,3$	$3,3$
	B	$3,2$	$2,3$	$1,2$
	C	$1,3$	$3,1$	$1,0$

There is one pure strategy Nash equlibrium: (A, C). Also, column B weakly dominates column C. If we remove column C, then row B dominates row A. If we then remove row A, we get the matrix from part (a), which does not have any pure strategy Nash equilibria.

12. There are four pure strategy Nash equilibria: (A, b, α), (B, a, α), (B, a, γ), and (A, b, γ).

13. Suppose that there is a linear utility function u representing this player's preferences. Then, from the preferences, we know:

$$u(A) > u(B) > u(C)$$

$$0.2u(A) + 0.7u(B) + 0.1u(C) > 0.3u(A) + 0.6u(B) + 0.1u(C)$$

Simplifying the second inequality, we get:

$$0.1u(B) > 0.1u(A)$$

This implies that $u(B) > u(A)$, which contradicts the previous inequality that $u(A) > u(B)$. Thus, there is not a linear utility function representing this player's preferences.

14. **Theorem.** *Suppose that the functions u and v are both linear utility functions that represent a certain player's preferences over compound lotteries on a finite set of outcomes. Then, the function $u + v$ is also a linear utility function that represents the player's preferences.*

Proof. Since u and v are both linear utility functions representing the player's preferences, v is a positive affine transformation of u. Thus, there exists $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $v = \alpha u + \beta$. Then:

$$u + v = u + \alpha u + \beta = (1 + \alpha)u + \beta$$

Since, $1 + \alpha > 0$ and $\beta \in \mathbb{R}$, we have that $u + v$ is a positive affine transformation of u. Thus, $u + v$ is also a linear utility function representing the player's preferences. □

15. **Theorem.** *Let $\succsim$ be a preference relation on the set of outcomes O that is complete, reflexive, and transitive. Let $\precsim$ be a relation on the set O defined as $x \precsim y$ if and only if $y \succsim x$. Then, the relation $\precsim$ is complete, reflexive, and transitive.*

    Proof. First, we prove that $\precsim$ is complete. Let $x, y \in O$. Since $\succsim$ is complete we know that $x \succsim y$ or $y \succsim x$ (or both). If $x \succsim y$, then $y \precsim x$. If $y \succsim x$, then $x \precsim y$. Thus, the relation $\precsim$ is complete.

    Next, we prove that $\precsim$ is reflexive. Let $x \in O$. Since $\succsim$ is reflexive, we have that $x \succsim x$. Thus, we have that $x \precsim x$, so $\precsim$ is reflexive.

    Finally, we prove that $\precsim$ is transitive. Let $x, y, z \in O$. Suppose $x \precsim y$ and $y \precsim z$. Then, by the definition of $\precsim$, we have that $y \succsim x$ and $z \succsim y$. Then, since $\succsim$ is transitive, we have that $z \succsim x$. Thus, we have that $x \precsim z$. Thus, the relation $\precsim$ is transitive. □