Math 315 Homework 4

Due Friday, March 3

Solutions must be written in LaTeX. You are encouraged to work with others on the assignment, but you should write up your own solutions independently. You should reference all of your sources, including your collaborators.

1. Suppose that a person whose preferences satisfy the von Neumann-Morgenstern axioms, and who always prefers more money to less money says that:

   - He is indifferent between receiving \$500 and participating in a lottery in which he receives \$1000 with probability 2/3 and receives \$0 with probability 1/3.
   - He is indifferent between receiving \$100 and participating in a lottery in which he receives \$500 with probability 3/8 and and receives \$0 with probability 5/8.

   (a) Find a linear utility function u representing this person's preferences and also satisfying $u(\$0) = 0$ and $u(\$1000) = 1$. What is $u(\$500)$ and $u(\$100)$?

   (b) Find a linear utility function v representing this person's preferences and also satisfying $v(\$0) = 3$ and $v(\$1000) = 8$. What is $v(\$500)$ and $v(\$100)$?

   (c) Determine which of the following two lotteries would be preferred by this person:

   $$L_1 = \left[\tfrac{3}{10}(\$1000), \tfrac{1}{10}(\$500), \tfrac{1}{2}(\$100), \tfrac{1}{10}(\$0)\right]$$

   $$L_2 = \left[\tfrac{2}{10}(\$1000), \tfrac{3}{10}(\$500), \tfrac{2}{10}(\$100), \tfrac{3}{10}(\$0)\right]$$

   (d) Does your answer to part (c) depend on whether you use the utilities from part (a) or part (b)?

   (e) Is it possible to determine whether the person would prefer to receive \$400 or participate in lottery L_1 (where L_1 is the lottery from part (c))? Justify your answer.

   (f) Is it possible to determine whether the person would prefer to receive \$600 or participate in lottery L_1 (where L_1 is the lottery from part (c))? Justify your answer.

2. (a) Prove that if v is a positive affine transformation of u, then u is a positive affine transformation of v.

   (b) Prove that if v is a positive affine transformation of u, and if w is a positive affine transformation of v, then w is a positive affine transformation of u.

3. A farmer wishes to dig a well in a square field whose vertices have coordinates $(0, 0)$, $(0, 1000)$, $(1000, 0)$ and $(1000, 1000)$. The well must be located at a point whose coordinates (x, y) are integers. The farmer's preferences are lexicographic: if $x_1 > x_2$, he prefers that the well be dug at the point (x_1, y_1) to the point (x_2, y_2) for all y_1, y_2. If $x_1 = x_2$, he prefers the first point only if $y_1 > y_2$. Give an example of a utility function representing such a preference relation. (Since no information is provided about the preference relation on lotteries, there are many possible utility functions.)

4. In this exercise, we will consider the situation in problem 3, but we will now allow the well to be located at any point (x, y) in the square (with x and y being real numbers). We will show that in this case, it is not possible to have a preference relation that satisfies the von Neumann-Morgenstern axioms (and thus, we are showing that when the set of possible outcomes is uncountable, it is not necessarily possible to have a preference relation that satisfies the axioms).

   The proof will use contradiction, so first, we assume that there does exist a preference relation over lotteries of points in the square that satisfies the von Neumann-Morgenstern axioms.

   (a) First, prove the following statement (assuming that there is a preference relation satisfying the axioms):

   For all (x, y) in the square there exists a unique number $p_{x,y}$ with $0 \leq p_{x,y} \leq 1$ such that the farmer is indifferent between locating the well at the point (x, y) and a lottery in which the well is located at point $(0, 0)$ with probability $1 - p_{x,y}$ and located at point $(1000, 1000)$ with probability $p_{x,y}$.

   (b) Prove that the function $f : (x, y) \to p_{x,y}$ is one-to-one. (That is, prove that $f(x_1, y_1) = f(x_2, y_2)$ implies that $(x_1, y_1) = (x_2, y_2)$.)

   (c) For each x with $0 \leq x \leq 1000$, define $A_x = \{p_{x,y} : 0 \leq y \leq 1000\}$. Prove that for each x the set A_x contains at least two elements.

   (d) Prove that the sets A_x are pairwise disjoint. (That is, for all x_1, x_2, we have that $A_{x_1} \cap A_{x_2} = \emptyset$.)

   (e) Prove that if $x_1 < x_2$ then for all $p_1 \in A_{x_1}$ and $p_2 \in A_{x_2}$, we have that $p_1 < p_2$. (There do not exist sets A_x satisfying (c), (d), (e); thus, we have a contradiction. Proving that such sets do not exist is a little complicated, so we will not prove it in this exercise, but if you think about it, it should make intuitive sense that such sets cannot exist.)