Math 316: Answers to Practice Problems

1. (a) $f = (1245)(3)(6)$ and $g = (16)(25)(34)$
   (b) $fg = (156)(234)$
   (c) $gf = (162)(354)$

2. $2(n-2)(n-3)!$

3. Let $n \in \mathbb{N}$. Then:
$$c(n, n-2) = \frac{1}{2}\binom{n}{2}\binom{n-2}{2} + 2\binom{n}{3}$$

   Proof. We know that $c(n, n-2)$ counts the number of permutations on n elements that have $n-2$ cycles. These $n-2$ cycles would either consist of two 2-cycles and $n-4$ singletons or one 3-cycle and $n-3$ singletons.

   There are $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ ways to create two 2-cycles: $\binom{n}{2}$ chooses the 2 elements of one 2-cycle; $\binom{n-2}{2}$ chooses the 2 elements for the second 2-cycle, and then we divide by 2, since the order of the 2-cycles does not matter.

   There are $2\binom{n}{3}$ ways to create the 3-cycle: there are $\binom{n}{3}$ ways to choose the 3 elements a, b, and c of the 3-cycle, and then there are two possible 3-cycles from those three elements, (abc) or (acb).

   Therefore, there are a total of $\frac{1}{2}\binom{n}{2}\binom{n-2}{2} + 2\binom{n}{3}$ permuations on n elements that have $n-2$ cycles. □

4. $3 - 3x + x(x-1) + 2x(x-1)(x-2)$

5. $1000 - (250 + 200 + 166) + (50 + 83 + 33) - 16 = 534$

6. $\dfrac{2x^2}{(1-x)^3}$

7. (a) $\dfrac{x}{(1-x)^2}$
   (b) $a_n = n$

8. (a) $\dfrac{1}{(1-x)^2(1+x)}$

(b) $a_n = \frac{1}{4} + \frac{1}{2}(n+1) + \frac{1}{4}(-1)^n$

9. (a) $\frac{1+x}{1-x-6x^2}$

(b) $a_n = \frac{4}{5} \cdot 3^n + \frac{1}{5} \cdot (-2)^n$

10. $\frac{1}{1-2x} + \frac{x}{(1-2x)(1-x)^2}$

11. (a) $\frac{1}{1-x}$

(b) e^{-x}

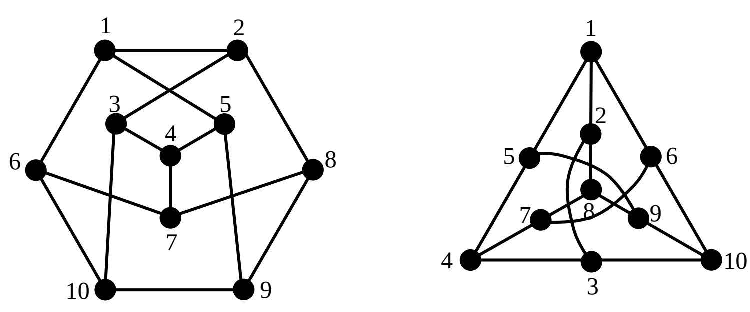
(c) e^{2x}

12. (a) $a_n = (-1)^n + 3 \cdot 2^n$

(b) $b_n = 3^n \cdot n!$

13. (a) The graphs are not isomorphic. In the first graph, all vertices have degree four or less, and the second graph has a vertex of degree 5.

(b) The two graphs are isomorphic, as shown in the following labeling:



14. $\frac{n!}{2}$ (Note that all such graphs are paths.)

15. (a) $$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

(b) 75