Math 316: Answers to Practice Problems

1. There are $24 \cdot 60 = 1440$ minutes in a day. With 1500 takeoffs, by the Pigeonhole Principle, there must be one minute during which two planes are taking off.

2. **Theorem.** Let $a_0 = 3$ and let $a_{n+1} = \sqrt{a_n + 7}$ if $n > 0$. Then, $3 < a_n < 4$ for all $n > 0$.

   Proof. We will prove this by induction on n.

   Base Case: We have that $a_1 = \sqrt{10}$. Then $3 < \sqrt{10} < 4$.

   Induction Step: Assume that for some $n > 0$, we have that $3 < a_n < 4$. Since $3 < a_n$, we have that $10 < a_n + 7$, and then $\sqrt{10} < \sqrt{a_n + 7}$. Thus, $\sqrt{10} < a_{n+1}$. Since $3 < \sqrt{10}$, we have that $3 < a_{n+1}$. Similarly, since $a_n < 4$, we have that $a_n + 7 < 11$, and then $\sqrt{a_n + 7} < \sqrt{11}$. Thus, $a_{n+1} < \sqrt{11}$. Since $\sqrt{11} < 4$, we have that $a_{n+1} < 4$.

   Thus, by induction $3 < a_n < 4$ for all $n > 0$. □

3. (a1) $3^4 = 81$

   (a2) $3 \cdot \dfrac{4!}{2!\,1!\,1!} = 36$

   (b) $9 \cdot 8 \cdot 7 \cdot 1 + 8 \cdot 8 \cdot 7 \cdot 1 = 952$

4. $\dfrac{9!}{1!\,2!\,1!\,2!\,3!} = 15{,}120$

5. $\dfrac{7!}{2!\,2!\,1!\,1!\,1!} - 2 \cdot \dfrac{6!}{2!\,2!\,1!\,1!} = 900$

6. $2^5 = 32$

7. (a) $\dbinom{6+15-1}{15} = \dbinom{20}{15} = 15{,}504$

   (b) $\dbinom{6+9-1}{9} = \dbinom{14}{9} = 2002$

   (c) $\dbinom{6+12-1}{12} = \dbinom{17}{12} = 6188$

8. (a) $\dfrac{\binom{9}{5}}{\binom{12}{5}} = \dfrac{126}{792} = \dfrac{63}{396}$

   (b) $\dfrac{\binom{4}{3}\binom{8}{2}}{\binom{12}{5}} = \dfrac{112}{792} = \dfrac{14}{99}$

   (c) $\dfrac{\binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4} + \binom{5}{2}\binom{7}{3}}{\binom{12}{5}} = \dfrac{546}{792} = \dfrac{273}{396}$

9. (a) $\dfrac{4}{36} = \dfrac{1}{9}$

   (b) $\dfrac{11}{36}$

   (c) $\dfrac{30}{36} = \dfrac{5}{6}$

10. (a) $\dbinom{12}{5} 3^5(-2)^7 = -24{,}634{,}368$

    (b) $\dbinom{6}{3,2,1} 2^3(-1)^2 = \dfrac{6!}{3!\,2!\,1!} \cdot 2^3(-1)^2 = 480$

    (c) $\dbinom{10}{3,7,0} = \dfrac{10!}{3!\,7!} = 120$

11. $1 - \dfrac{4}{5}x - \dfrac{2}{25}x^2 - \dfrac{4}{125}x^3 - \dfrac{11}{625}x^4$

12. **Theorem.** Let $k, n \in \mathbb{N}$ with $n \geq k$. Then,

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Proof. Both sides count the number of ways to choose a committee of k people from n people and to choose a president of the committee (with the president being one of the people on the committee).

On the left-hand side, we first choose the committee (there are $\binom{n}{k}$ ways to choose the k people to form the committee), and then we choose the president from among the k

people on the committee (so there are k choices for the president of the committee). Thus, there are $k\binom{n}{k}$ ways to choose the committee and the president.

On the right-hand side, we first choose the president from the n people (there are n ways to do this), and then we choose the remaining $k-1$ people on the committee from among the $n-1$ remaining people (there are $\binom{n-1}{k-1}$ ways to do this). Thus, there are $n\binom{n-1}{k-1}$ ways to choose the committee and the president.

Therefore, $k\binom{n}{k} = n\binom{n-1}{k-1}$. □

13. **Theorem.** Let $n \in \mathbb{N}$. Then,
$$3^n = \sum_{k=1}^{n} \binom{n}{k} 2^k$$

Proof. Recall the Binomial Theorem:
$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k}$$

Let $x = 2$ and $y = 1$. Then:
$$3^n = \sum_{k=1}^{n} \binom{n}{k} 2^k$$

□

14. $2^{13} = 8192$

15. $S(4,1) + 2 \cdot S(4,2) + 3 \cdot S(4,3) + 4 \cdot S(4,4) = 1 + 7 \cdot 2 + 6 \cdot 3 + 4 \cdot 1 = 37$

16. Let $n \in \mathbb{N}$. Then,
$$S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$$

Proof. We know that $S(n, n-2)$ counts the number of partitions of the set $\{1, 2, 3, \ldots, n\}$ into $n-2$ blocks. When we partition $\{1, 2, 3, \ldots, n\}$ into $n-2$ blocks, we could either partition the set into $n-3$ singletons and one set with three elements, or we could partition the set into $n-4$ singletons and two doubletons. There are $\binom{n}{3}$ ways to partition into $n-3$ singletons and one set with three elements (since you need to choose the three elements to be in one set).

To partition into $n-4$ singletons and two doubletons, we need to first choose the four elements to be in the doubletons (there are $\binom{n}{4}$ ways to choose these), and then we need to decide how to divide up the four elements into doubletons (there are three ways to partition four elements into doubletons: $\{a, b\}\{c, d\}$, $\{a, c\}\{b, d\}$, and $\{a, d\}\{b, c\}$). Thus, there are $3\binom{n}{4}$ ways to partition into $n-4$ singletons and two doubletons.

Therefore, there are a total of $\binom{n}{3} + 3\binom{n}{4}$ ways to partition the set $\{1, 2, 3, \ldots, n\}$ into $n-2$ blocks. $\square$