

Math 316 Homework 10

Due Friday, May 6

Solutions should be written neatly and legibly (and do not need to be written in \LaTeX). You are encouraged to work with others on the assignment, but you should write up your own solutions independently. You should reference all of your sources, including your collaborators.

1. Let D_n be the number of derangements of $\{1, 2, 3, \dots, n\}$ with $D_0 = 1$ and $D_1 = 0$.

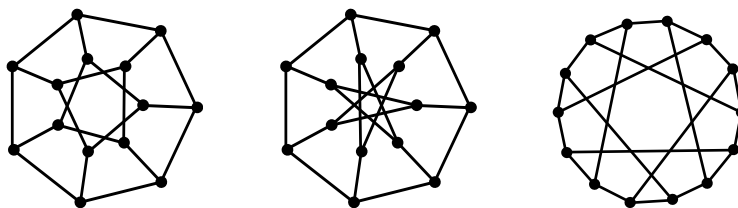
(a) Let $n \geq 1$. Prove the following identity:

$$D_n = nD_{n-1} + (-1)^n$$

(*Hint:* Use the inclusion-exclusion formula for derangements.)

(b) Let $D(x)$ be the exponential generating function for D_n . Use the recursive formula from part (a) to find a closed form for $D(x)$.

2. Determine which of the following graphs are isomorphic. Prove your answer.



3. An **automorphism** ϕ of a simple graph G is a one-to-one mapping of the vertex set of G onto itself with the property that $\phi(v)$ and $\phi(w)$ are adjacent whenever v and w are adjacent (in other words, an automorphism is an isomorphism from G to G).

- (a) How many automorphisms does K_4 have?
- (b) How many automorphisms does $K_{3,3}$ have?
- (c) Construct a simple graph with six vertices whose only automorphism is the identity.
- (d) Construct a simple graph that has exactly three automorphisms. (*Hint:* Think of a rotating triangle with appendages to prevent flips.)

4. (Exercise 30, Chapter 9) Prove that in any simple graph, there are two vertices with the same degree.

Extra Credit:

The *girth* of a graph is the length of the shortest cycle. Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k then G has at least $k^2 + 1$ vertices.