

Math 316 Homework 4

Due Friday, March 4

Solutions must be written in L^AT_EX. You are encouraged to work with others on the assignment, but you should write up your own solutions independently. You should reference all of your sources, including your collaborators.

1. Let n , p , and q be fixed positive integers with $p \leq q \leq n$. Prove the following identity:

$$\binom{n}{p} \binom{n}{q} = \sum_{k=0}^p \binom{n}{k} \binom{n-k}{p-k} \binom{n-p}{q-k}$$

2. (Exercise 51, Chapter 4) Consider the expression $(x_1 + x_2 + \cdots + x_k)^n$.

- (a) Suppose that when we expand this power, there is an integer that occurs as a coefficient only once. What relation does that imply between k and n ?
- (b) Can it happen that there will be more than one coefficient that occurs only once in the expansion? If it can, given an example. Otherwise, explain why not.

3. (a) Consider the following function:

$$f(x) = \frac{-1}{\sqrt{1-x^2}}$$

Use Newton's Binomial Theorem to find the first four nonzero terms of the power series for $f(x)$.

- (b) Take the integral of your answer to part (a) to find the first four nonzero terms of the power series for $g(x) = \cos^{-1} x$.

4. (a) Consider the following function:

$$f(x) = \frac{x}{(1-x)^2}$$

Find the power series for $f(x)$. (*Hint:* It works to simply multiply the power series for $1/(1-x)^2$ by x .)

- (b) Use your answer to part (a) to find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

(c) Find a function $g(x)$ that has power series $\sum_{k=1}^{\infty} k(k-1)x^k$.

(d) Use your answer to part (c) to find the sum of the series $\sum_{n=1}^{\infty} \frac{n(n-1)}{2^n}$.

Extra Credit:

Find a closed formula for the sum of every fourth term of the n th row of Pascal's triangle:

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k}$$

(*Hint:* Try using imaginary numbers in the Binomial Theorem. Your answer may involve i .)