Math 316 Homework 5

Due Friday, March 11

Solutions must be written in LaTeX. You are encouraged to work with others on the assignment, but you should write up your own solutions independently. You should reference all of your sources, including your collaborators.

1. We want to select as many subsets of $\{1, 2, 3, \ldots, n\}$ as possible without selecting two subsets where one subset contains the other. Prove that we can select at least $2^n/n$ subsets that do this.

2. Prove that the number of 2-by-n arrays

$$\begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \end{bmatrix}$$

that can be made from the numbers $1, 2, \ldots 2n$ such that each number is used exactly once and

$$x_{11} < x_{12} < \cdots < x_{1n}$$

$$x_{21} < x_{22} < \cdots < x_{2n}$$

$$x_{11} < x_{21}, x_{21} < x_{22}, \ldots, x_{1n} < x_{2n}$$

equals the nth Catalan number C_n.

3. (Exercise 37, Chapter 4) What is the number of northeastern lattice paths from $(0, 0)$ to (n, n) that never touch the main diagonal other than in the starting and ending point? Prove your answer.

4. Suppose that Anna and Bill are candidates for some political office and that Anna receives a votes and Bill receives b votes with $a > b$. The votes are counted one at a time. Suppose that as the votes are counted, Anna is always tied with Bill or ahead of Bill (meaning that after each vote is counted, the number of votes counted so far for Anna is greater than or equal to the number of votes counted so far for Bill). Show that the probability of this occurring is $\dfrac{a - b + 1}{a + 1}$.

Extra Credit

Let $n = 3k$. Prove that

$$\lim_{n \to \infty} \frac{\displaystyle\sum_{i=0}^{k} \binom{n}{3i}}{2^n} = \frac{1}{3}$$

In other words, the sum of every third element of the nth row of Pascal's triangle is roughly one third of the sum of all elements of that row. (*Hint:* Use a similar method to the Extra Credit from Homework 4 to find a formula for the summation.)