Math 316 Homework 7

Due Friday, April 8

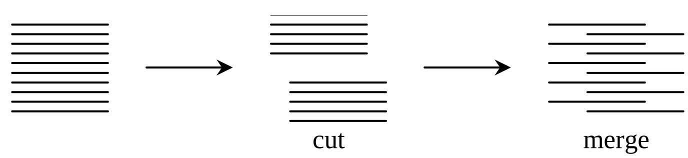
Solutions must be written in LaTeX. You are encouraged to work with others on the assignment, but you should write up your own solutions independently. You should reference all of your sources, including your collaborators.

1. Let $F(n)$ be the number of all partitions of $[n]$ with no singleton blocks.
   (a) Find $F(2)$, $F(3)$, and $F(4)$.
   (b) (Exercise 33, Chapter 5) Find a recursive formula for the numbers $F(n)$ in terms of the numbers $F(i)$ with $i \leq n-1$.

2. Let n be a positive integer. Prove the following identity:

$$(-1)^n = \sum_{k=0}^{n} (-1)^k k! S(n,k)$$

3. In a **perfect riffle shuffle**, a deck of 52 cards is cut into two halves, which are then merged in an interleaving fashion:



   Note that the top card of the deck remains on top after the shuffle.
   (a) Write a permutation $p : [52] \to [52]$ that represents a perfect riffle shuffle. Write p using cycle notation.
   (b) What is p^8? In simple terms, what does this say about perfect riffle shuffles?

4. Consider the polynomial $1 - 3x + 2x^2 - x^3 - 2x^4$. Write this polynomial as a linear combination of the polynomials 1, x, $x(x-1)$, $x(x-1)(x-2)$, and $x(x-1)(x-2)(x-3)$.

Extra Credit

Let $a(n,k)$ be the number of permutations of length n with k cycles in which the entries 1 and 2 are in the same cycle. Prove that for $n \geq 2$:

$$\sum_{k=1}^{n} a(n,k)x^k = x(x+2)(x+3)\cdots(x+n-1)$$