

Math 316 Homework 8

Due Friday, April 15

Solutions must be written in L^AT_EX. You are encouraged to work with others on the assignment, but you should write up your own solutions independently. You should reference all of your sources, including your collaborators.

1. (a) Determine the number of positive integers $k \leq 15,000$ that are divisible by 3, 4, 5, or 10.
(b) Determine the number of positive integers $k \leq 15,000$ that are neither perfect squares nor perfect cubes.
2. A subway has six stops on its route from its base location. There are 10 people on the subway as it departs its base location. Each person exits the subway at one of its six stops, and at each stop at least one person exits. In how many ways can this happen? (Your answer should be a number.)
3. (Exercise 20, Chapter 7) Let $D(n)$ be the number of derangements of $\{1, 2, 3, \dots, n\}$ with $D(0) = 1$ and $D(1) = 0$. Let $n \geq 1$. Give a combinatorial proof of the following identity:

$$D(n+1) = n(D(n) + D(n-1))$$

Do not use the formula for $D(n)$ that we proved in class.

4. For this problem, you will use Theorem 7.6 from the textbook, which we did not cover in class. Let $X = \{1, 2, 3, \dots, n\}$, and let $\mathcal{P}(X)$ be the power set of X . Suppose that $f : \mathcal{P}(X) \rightarrow \mathbb{R}$ and $g : \mathcal{P}(X) \rightarrow \mathbb{R}$ are functions so that for all $S \subseteq X$:

$$g(S) = \sum_{T \subseteq S} f(T)$$

Also, suppose that $g(S) = 4^{|S|}$. Then, Theorem 7.6 will give a formula for $f(S)$ as a summation. Use this to answer the following questions.

- (a) If $X = \{1\}$, what is $f(X)$?
- (b) If $X = \{1, 2\}$, what is $f(X)$?
- (c) If $X = \{1, 2, 3\}$, what is $f(X)$?
- (d) Conjecture a simple closed formula (not involving a summation) for $f(X)$, and then prove your conjecture. (*Hint:* Use the Binomial Theorem.)

Extra Credit:

Let $D(n)$ be the number of derangements of $\{1, 2, 3, \dots, n\}$. Prove the following identity for positive integers k and n with $k \leq n$:

$$\sum_{i=0}^k \binom{k}{i} D(n-i) = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)!$$