Math 316: Practice Problems for Final Exam

1. Consider the following permutations $f : [6] \to [6]$ and $g : [6] \to [6]$:

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 1 & 6 \end{bmatrix} \qquad g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

   (a) Write f and g using cycle notation.

   (b) Find fg and write it using cycle notation.

   (c) Find gf and write it using cycle notation.

2. Consider permutations $p : [n] \to [n]$. How many of these permutations have 1 and 2 as part of the same 3-cycle?

3. Show that the signless Stirling numbers of the first kind satisfy the following relationship:

$$c(n, n-2) = \frac{1}{2}\binom{n}{2}\binom{n-2}{2} + 2\binom{n}{3}$$

4. Consider the polynomial $3 - 5x^2 + 2x^3$. Write this polynomial as a linear combination of the polynomials 1, x, $x(x-1)$, $x(x-1)(x-2)$.

5. Find the number of integers between 1 and 1,000 inclusive that are not divisible by 4, 5, or 6.

6. Determine the generating function for the sequence $a_n = n(n-1)$.

7. Suppose we are creating bags of fruit containing apples, oranges, bananas, and pears, and suppose that each bag must contain an even number of apples, at most one orange, an odd number of bananas, and at most one pear. Let a_n equal the number of such bags containing n pieces of fruit.

   (a) Find the generating function for a_n.

   (b) Find a closed formula for a_n.

8. Let a_n be the number of possible values for x_1 and x_2 that satisfy

$$x_1 + x_2 = n$$

with the conditions that x_1 and x_2 are non-negative integers and x_1 is even.

(a) Find the generating function for a_n.

(b) Find a closed formula for a_n.

9. Let $a_0 = 1$, $a_1 = 2$, and for $n \geq 2$ let a_n satisfy the following recurrence relation:

$$a_n = a_{n-1} + 6a_{n-2}$$

(a) Find the generating function for a_n.

(b) Find a closed formula for a_n.

10. Let $b_0 = 1$ and for $n \geq 1$ let b_n satisfy the following recurrence relation:

$$b_n = 2b_{n-1} + n$$

Find the generating function for b_n.

11. Determine the exponential generating functions for the following sequences:

(a) $0!, 1!, 2!, 3!, 4!, 5!, \ldots, n!, \ldots$

(b) $1, -1, 1, -1, 1, -1, 1, -1, \ldots, (-1)^n, \ldots$

(c) $1, 2, 4, 8, 16, 32, \ldots, 2^n, \ldots$

12. (a) Suppose that a_n is a sequence and suppose that the exponential generating function for a_n is:

$$A(x) = e^{-x} + 3e^{2x}$$

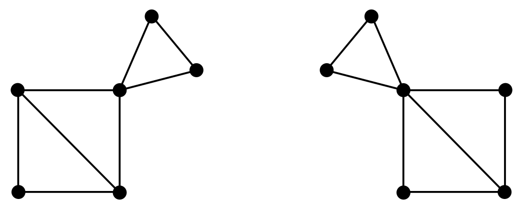
Find a closed formula for a_n.

(b) Suppose that b_n is a sequence and suppose that the exponential generating function for b_n is:

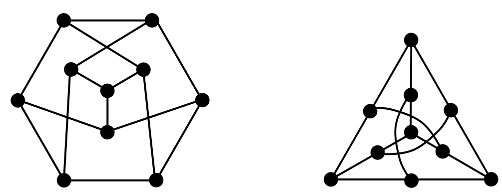
$$B(x) = \frac{1}{1 - 3x}$$

Find a closed formula for b_n.

13. (a) Determine whether the following graphs are isomorphic. Justify your answer.

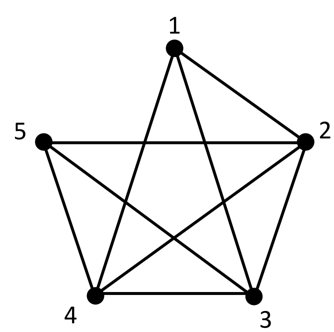


(b) Determine whether the following graphs are isomorphic. Justify your answer.



14. Consider labelled trees in which every vertex has degree 1 or 2. How many such labelled trees are there on n vertices?

15. Consider the following graph:



(a) Find the Laplacian matrix for this graph.

(b) How many labelled spanning trees does this graph have? (*Hint:* Use row reduction to compute the determinant.)