Math 316: Practice Problems for Midterm Exam

1. (Exercise 1, Chapter 1) A busy airport sees 1500 takeoffs per day. Prove that there are two planes that must take off within a minute of each other.

2. (Exercise 27, Chapter 2) Let $a_0 = 3$ and let $a_{n+1} = \sqrt{a_n + 7}$ if $n > 0$. Prove that $3 < a_n < 4$ for all $n > 0$.

3. (a1) How many four-digit numbers contain only the digits 3, 5, and 9?

   (a2) How many four-digit numbers contain only the digits 3, 5, and 9, with each of those digits appearing at least once?

   (b) How many four-digit numbers (where zero cannot be the first digit) are divisible by 5 and have no repeated digits?

4. How many permutations are there of the word ADDRESSES?

5. How many ways are there to list the digits 1, 1, 2, 2, 3, 4, 5 so that the digits 3 and 4 are not in consecutive positions?

6. Consider them multiset $\{1, 1, 1, 1, 1, 2, 3, 4, 5, 6\}$. Determine the number of 5-element sub-multisets of this multiset.

7. A bagel store sells six different kinds of bagels.

   (a) How many ways can you choose 15 bagels?

   (b) How many ways can you choose 15 bagels, including at least one of each kind?

   (c) If one of the kinds is Sesame, how many ways can you choose 15 bagels, including at least three Sesame bagels?

8. A bag contains 3 red marbles, 4 green marbles, and 5 blue marbles. A sample of 5 is randomly selected from the box.

   (a) What is the probability that the sample contains no red marbles?

   (b) What is the probability that the sample contains exactly 3 green marbles?

   (c) What is the probability that the sample contains at most 2 blue marbles?

9. Two standard dice are rolled.

  (a) What is the probability that the sum of the rolls is 5?

  (b) What is the probability that one or both of the die show a 3?

  (c) What is the probability that the two numbers rolled are different?

10. (a) What is the coefficient of $x^5 y^7$ in the expansion of $(3x - 2y)^{12}$?

  (b) What is the coefficient of $x^3 y^2 z$ in the expansion of $(2x - y + z)^6$?

  (c) What is the coefficient of $x^3 y^7$ in the expansion of $(x + y + z)^{10}$?

11. Find the first five terms of the Taylor series for $f(x) = (1 - x)^{4/5}$.

12. Find a **combinatorial proof** of the following identity:

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

  (Note: We want a combinatorial proof, not an algebraic proof.)

13. Use the binomial theorem to prove the following identity:

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

14. (Exercise 17, Chapter 5) How many compositions does the integer 15 have whose first part is not 1?

15. How many partitions does the set $\{1, 2, 3, 4, 5\}$ have in which the element 1 is not in a block by itself?

16. Prove that the Stirling numbers of the second kind satisfy the following relationship:

$$S(n, n-2) = \binom{n}{3} + 3 \binom{n}{4}$$