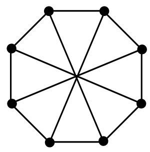
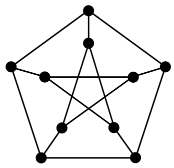
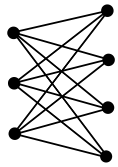
Math 317 Final Exam

Due Friday, December 19 by 5pm

You must work completely on your own, consulting only the textbook, your course notes, and your homeworks as references. Show all of your work. If you have questions, you can come to my office hours or ask me via e-mail. You should write your solutions neatly and legibly, and justify all of your answers. Good luck!

1. A graph G is **1-Hamiltonian** if for all vertices $v \in V(G)$, the graph $G - v$ (obtained by deleting v and the edges incident to v) is Hamiltonian.

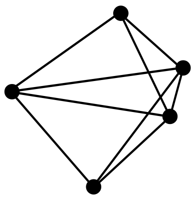
   (a) **[15 pts.]** For each of the following graphs, determine whether the graph is 1-Hamiltonian. Justify your answers.

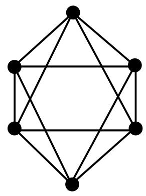
   
   
   

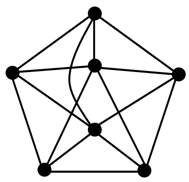
   (b) **[6 pts.]** Find a graph that is Hamiltonian but not 1-Hamiltonian.

   (c) **[9 pts.]** Let G be a 1-Hamiltonian graph and let u and v be two distinct vertices in G. Prove that there are at least three disjoint paths from u to v.

2. **[15 pts.]** The *bipyramid* B_n is the graph with n vertices consisting of a cycle with $n-2$ vertices and 2 additional vertices that are both adjacent to all of the vertices in the cycle. The bipyramids B_5, B_6, and B_7 are shown below:

   
   B_5

   
   B_6

   
   B_7

   Determine the chromatic polynomial of B_n for $n \geq 5$. Prove your answer.

3. **[20 pts.]** Let G be a simple bipartite graph in which every vertex has degree r.

   (a) Use Hall's Theorem to show that G has a complete matching.

   (b) Without using Theorem 20.4 from Edition 4 of the textbook or Theorem 5.18 from Edition 5 of the textbook, prove that the chromatic index of G is r. (*Hint:* Use part (a).)

4. **[15 pts.]** Let G be a simple bipartite graph with bipartition A and B in which every vertex in A has degree at least t with $t > |A|$, and suppose that G satisfies the conditions of Hall's Theorem (every subset $X \subseteq A$ is adjacent to at least $|X|$ vertices in B). Since $t > |A|$, the graph G will have multiple complete matchings from A to B. Use induction on $|A|$ to show that G has at least $\dfrac{t!}{(t-|A|)!}$ complete matchings from A to B.

5. **[20 pts.]** Consider the matroid M with $E = \{a, b, c, d, e\}$ and with independent sets

$$\mathcal{I} = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, e\}, \{d, e\}\}$$

   (a) Is this matroid isomorphic to a graphic matroid? If so, show the corresponding graph. If not, explain why not.

   (b) Is this matroid isomorphic to a representable matroid over $\mathbb{R}^n$? If so, show a set of vectors for the representable matroid. If not, explain why not.