Graph Theory and the Game of Sprouts

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This article concerns a game that children can play. As described by Martin Gardner [1] it was invented and studied by John H. Conway and Michael S. Paterson. This account has written it as a digression on Euler's formula that one might present to an undergraduate graph theory class.

Sprouts is a game played with pen and paper. In it the players begin with a finite set of m points. For his turn a player adds both an edge and a vertex to the graph in a prescribed way. Namely, the player must first add an edge between two extant vertices (or add a loop at a single vertex) and then subdivide the edge he just added with a new vertex. To make the game finite, we require that no vertex ever be of degree greater than 3, and to make it interesting, we prohibit edges from crossing one another. The winner of the game is, say, the player who makes the last possible move.

In what follows we consider upper and lower bounds on the number of plays in a complete game of Sprouts on m points. In the first proposition, it is shown that a game may last $3m - 1$ plays but no more. In Propositions 2 and 3 it is shown that a game may last $2m$ plays but no less. It is shown in Proposition 5 that a game whose final graph is connected requires strictly more than $2m$ plays while in Proposition 4 it is shown that if the final graph is 2-connected, the number of plays required is not less than $(7m - 2)/3$.

There is only one game possible if $m = 1$. Although such games may be drawn in apparently different ways (see FIGURE 1), each can be identified with a game

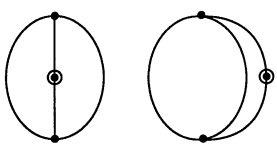

Figure 1

played on a sphere, this by identifying the plane (homeomorphically) with a punctured sphere. Once on the sphere any game on one point can be deformed to any other. There are two different games on two points shown in FIGURE 2. Note that the game in (2a) has stopped because there is no legal way to connect the two vertices of degree 2 (the circled vertices) without crossing an edge. As we will see,

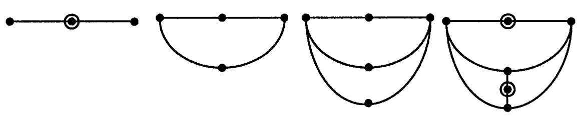

Figure 2a

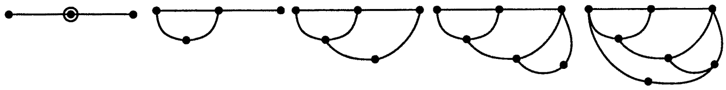

Figure 2b

these two games are extremal in the sense that every game on two points must last at least four moves and no more than five.

Proposition 1. *A game on m points can last no longer than* $3m - 1$ *plays, and a game of that length is always possible.*

Proof: Suppose a game starts with m points. After p plays there are $m + p$ vertices, and since each play adds four to the total degree of the graph, the total degree after p plays is $4p$. According to the rules of play, no vertex is allowed to have degree larger than 3. Hence after p plays the total degree will not exceed $3(m + p)$. Actually, it must be strictly less since there is always a vertex of degree 2 after play has started. Thus $4p < 3(m + p)$ or $p < 3m$.

We will now describe a game that takes $3m - 1$ plays to finish. First note that when a game on one point is played, the plane is divided into 3 regions, and the vertex of degree 2 lies on the boundary of precisely two of these regions. Now fix a point and play a game on that point so that after the two plays all the remaining points are contained in one of the regions with the degree 2 vertex in its boundary. This is the first step. For the second, play the one point game at a second vertex in such a way that all the remaining degree zero vertices as well as the first one-game lie in a single region with the degree 2 vertex of the second one-game in its boundary. With the next play, connect the two vertices of degree 2. See FIGURE 3. The "dumbbell" so constructed contains a single vertex of degree 2 and it lies in the boundary of the region containing the remaining null-graph. That is, we are effectively in the same situation as after the first step, and we can repeat the second step until all the points are used up. The first step took two plays and the second three. Hence the total will be $3m - 1$. This proves the proposition.

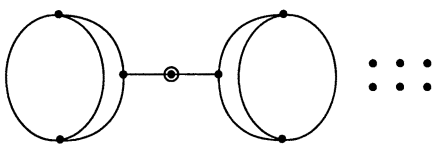

Figure 3

Proposition 2. *There is a complete game of Sprouts on m points which takes exactly* $2m$ *plays.*

Proof: As in the long game described in Proposition 1, choose a point and play a one-game on it. Keep the null-points in a single region, but this time put them in the region that does *not* contain the degree 2 vertex in its boundary. This isolates the degree 2 vertex, and it must remain degree 2 for the remainder of the game.

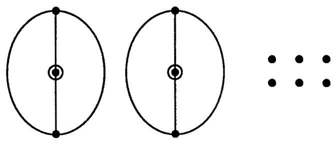

Figure 4

Repeat this step for each of the remaining points. See FIGURE 4. This process yields a game that terminates in $2m$ plays.

As we will explain, not only is this a game of minimal length, but every other game must last longer. First, however, let us recall Euler's formula for planar graphs. Let G be any graph; suppose it has e edges and v vertices. We say that G is planar if it can be drawn in the plane so that none of its edges cross. Since the 2-plane is homeomorphic to the 2-sphere with a single point removed, we may also think of a planar graph as one arising from some polyhedron in 3-space. The edges of a planar graph divide the plane into regions which, thinking of the associated polyhedron, we will call faces. Let f be the number of faces in G. Euler's formula relates these quantities:

$$f - e + v = 2.$$

The figures that arise from a game of sprouts are not always graphs in the standard sense of the word since there may be multiple edges between vertices as in FIGURE 1. In the next paragraph, we will modify these figures so that even loops occur. Nevertheless Euler's formula remains valid in each instance, and we shall continue to call the figures under consideration "graphs."

Suppose that a graph G_0 has been obtained from a complete game of sprouts. Each vertex of G_0 is of degree 2 or 3. This graph is a "subdivision" of a unique cubic graph G which we may obtain from G_0 as follows. Suppose x is a vertex of degree 2 and suppose that x is adjacent to y. Remove an edge connecting x and y, then identify the vertices x and y. Do this once at each degree 2 vertex. That is, we contract one edge incident to each degree 2 vertex. Although we obtain a cubic graph (i.e., each vertex is of degree 3), we don't want to forget where the degree 2 vertices were, so we color each of the edges which before had been incident to a degree 2 vertex; red, say. In the figures, a degree 2 vertex is to be understood as an indication that the two incident edges are to be considered as a single red edge. Note that this process of contraction can produce loops in G, but that such loops will always be colored red.

Lemma 1. *Suppose that the cubic graph G arises as just described from a complete game of Sprouts played on m vertices in p plays. Then*

$$f = 2 + p - m.$$

Proof: Let r be the number of red edges in G. Reasoning as in Proposition 1, the total degree of G_0 is $4p$, and it is also $3(m + p) - r$. In particular, $r = 3m - p$. The number of edges in G, e, is $2p - r = 3(p - m)$. The number of vertices, v, is $m + p - r = 2(p - m)$. The lemma follows by substituting these values for e and v into Euler's formula for G.

Recall that a graph is connected if there is a path between any pair of vertices, and that an edge in a connected graph is called a bridge if its removal disconnects the graph. The next proposition shows that the game in Proposition 2 is of minimal length.

Proposition 3. *Suppose that the cubic graph G arises from a complete game of Sprouts on m vertices in p plays. Then*

$$p \geq 2m.$$

Proof: We may assume that G is connected since the general case follows easily from this. Observe that no face in the graph G can have more than one red edge, for otherwise the generating game of Sprouts would not have been complete. Hence $f \geq r$. Since $r = 3m - p$, it follows from Lemma 1 that $2 + p - m \geq 3m - p$, and hence that

$$p \geq 2m - 1.$$

To finish the proof we must show that this inequality is strict. Suppose therefore that $p = 2m - 1$. Then, again by Lemma 1, $f = m + 1$. On the other hand, $r = m + 1$ as well. Thus each face must have a red edge and each red edge must lie in the boundary of a single face. Now an edge lies in the boundary of one or two edges according to whether or not it is a bridge. Thus each face in G has a bridge in its boundary. But this is impossible: G is not a tree since it is cubic and all trees have end vertices (vertices of degree 1). Thus G has at least one cycle. Since G is finite there must be a cycle with no other cycle in its interior. The interior of this cycle is a face of G which can have no bridge in its border without also having an end vertex. This proves the proposition.

What is the shortest game on m points whose final graph is connected? I don't know. The next two propositions give some information in this direction, however. In the first we glean a little more information from Euler's formula, and in the second we establish our claim that any game of $2m$ plays on m points must be the game described in Proposition 2. We say that a graph is 2-connected if it is connected and it contains no bridges.

Proposition 4. *Suppose that the cubic graph G arises from a complete game of Sprouts on m vertices in p plays. If G is 2-connected, then*

$$p \geq \tfrac{7}{3}m - \tfrac{2}{3}. \qquad (*)$$

Proof: Since each edge lies in the boundary of exactly two faces, there must be twice as many faces as red edges. Thus, since $f \geq 2r$, we obtain

$$2 + p - m \geq 2(3m - p),$$

which simplifies to $(*)$.

It remains for us to consider graphs which arise from a game of Sprouts which are connected but not necessarily 2-connected.

Proposition 5. *Suppose that the cubic graph G arises from a complete game of Sprouts on m vertices in p plays. If G is connected and $m > 2$, then*

$$p > 2m.$$

Proof: In light of Proposition 3 we need only show $p \neq 2m$. Suppose then that $p = 2m$. Then $f = 2 + m$ by Lemma 1. Let b be the number of red bridges in G. We have $m \geq b \geq m - 2$. If all the red bridges are removed from G, the connected components include at least two nontrivial subgraphs to which only one bridge was attached. Such subgraphs cannot be loops since loops must be red, and no two red edges can be adjacent. Hence we can contract by an edge incident to the vertex where the bridge was attached and obtain a cubic planar graph. In such graphs $f \geq 3$ since $2e = 3v$. In particular, there are at least two interior faces. Thus

$$f = m + 2 \geq b + 4,$$

and b is actually equal to $m - 2$. Moreover, there must be exactly two end components when the red bridges are deleted and each of these must have exactly two interior faces. In the original graph G such a component has four edges, exactly one of which is red. But if we recall how the game is played, we realize that there must be an even number of edges in any such component which are not colored. Consequently, this configuration is also impossible, and the proposition follows.

Given the nature of the subject it should be no surprise that more questions have been raised than settled. It would certainly be more satisfying to have sharp lower bounds on the number of plays in a connected or a 2-connected game. It would also be interesting to know what happens to such bounds if Sprouts is played on some other manifold since we have relied so heavily on the Euler characteristic $(f - e + v)$ of the sphere. One might also wonder which cubic planar graphs G can arise from a game of Sprouts. That is, when can G be decomposed into an edge sum of P_1 and P_2 subgraphs in such a way that no two of the P_1 summands bound the same face? (P_1 and P_2 denote the path graphs of one and two edges respectively.) Finally, since G is planar, it has a dual graph G^* in which the roles of vertices and faces are interchanged. In the context of G^*, we want to know when a set of edges is maximal with respect to both independence and the rules of the game. If G is 2-connected, for example, Tutte's 1-factor theorem applies to G^*, and gives a condition under which the inequality of Proposition 4 is strict.

REFERENCE

1. M. Gardner, *Mathematical Carnival*, Alfred A. Knopf, New York, 1975.

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"The sine curve of Bush's presidency was nearly as predictable as geometry, if his Campaign behaviour is taken as the axiom."

from an article "All the President's Wars"
by Sidney Blumenthal.
***New Yorker*, Dec. 28, 1992-Jan. 4, 1993, p. 66 lines 8–12.**

—Contributed by Emma Lehmer