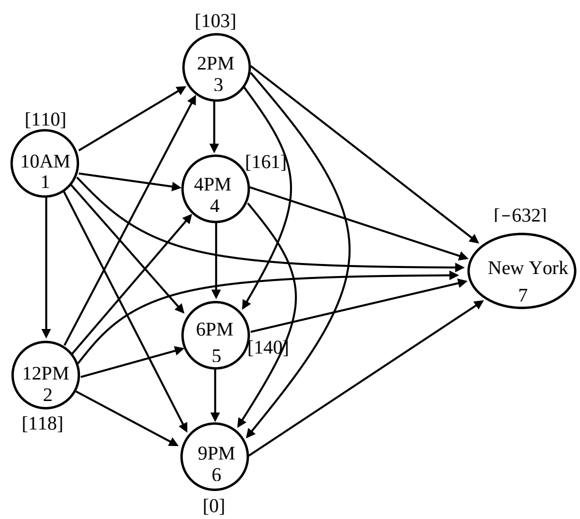
Math 322: Answers to Practice Problems for Final

1. (a) Assign Applicant A to Job 2, Applicant B to Job 3, and Applicant C to Job 1. The total cost is $15.

   (b) There are two optimal solutions:

   - Assign Applicant A to Job 1, Applicant B to Job 2, Applicant C to Job 4, and Applicant D to Job 3. The total cost is $19.
   - Assign Applicant A to Job 1, Applicant B to Job 3, Applicant C to Job 4, and Applicant D to Job 2. The total cost is $19.

2. (a) The network is:



The net flow through each vertex is labeled in the graph. A Minimum Cost Flow problem needs to also have two labels on each edge: the cost and the capacity. As it would be hard to put these labels on the above graph (since it is already so complicated), the costs and capacities are shown in the following tables:

Capacities

	10AM	12PM	2PM	4PM	6PM	9PM	New York
10AM	—	∞	∞	∞	∞	∞	100
12PM	—	—	∞	∞	∞	∞	100
2PM	—	—	—	∞	∞	∞	100
4PM	—	—	—	—	∞	∞	150
6PM	—	—	—	—	—	∞	150
9PM	—	—	—	—	—	—	∞
New York	—	—	—	—	—	—	—

Costs

	10AM	12PM	2PM	4PM	6PM	9PM	New York
10AM	—	\$100	\$200	\$300	\$400	\$625	\$0
12PM	—	—	\$100	\$200	\$300	\$525	\$0
2PM	—	—	—	\$100	\$200	\$425	\$0
4PM	—	—	—	—	\$100	\$325	\$0
6PM	—	—	—	—	—	\$225	\$0
9PM	—	—	—	—	—	—	\$0
New York	—	—	—	—	—	—	—

(b) The vertices in the network above are labeled 1 through 7. The variables are:

$$x_{ij} = \text{the number of people who move from vetex } i \text{ to } j$$

Also, let c_{ij} = the cost to United of moving one person from vertex i to j. Then, the linear program is

$$\text{Max } Z = \sum_{i,j} c_{ij} x_{ij}$$

subject to

$$x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} = 110$$
$$x_{23} + x_{24} + x_{25} + x_{26} + x_{27} - x_{12} = 118$$
$$x_{34} + x_{35} + x_{36} + x_{37} - x_{13} - x_{23} = 103$$
$$x_{45} + x_{46} + x_{47} - x_{14} - x_{24} - x_{34} = 161$$
$$x_{56} + x_{57} - x_{15} - x_{25} - x_{35} - x_{45} = 140$$
$$x_{67} - x_{16} - x_{26} - x_{36} - x_{46} - x_{56} = 0$$
$$-x_{17} - x_{27} - x_{37} - x_{47} - x_{57} - x_{67} = -632$$
$$x_{17} \leq 100, x_{27} \leq 100, x_{37} \leq 100, x_{47} \leq 150, x_{57} \leq 150$$
$$x_{ij} \geq 0$$

3. Let $x_i = \begin{cases} 1 & \text{if there is a Starbucks in region } i \\ 0 & \text{if there is no Starbucks in region } i \end{cases}$

Then, the following integer program will determine the optimal regions to locate the Starbucks.

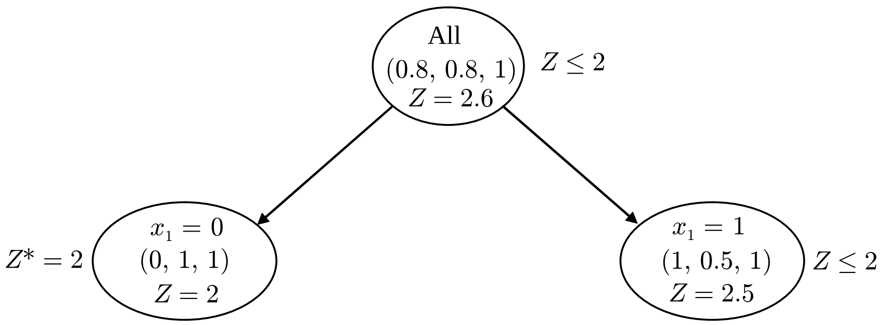
$$
\begin{aligned}
&\text{Max } \sum_{i=1}^{11} x_i \\
&\text{subject to} \\
&x_1 + x_2 + x_3 + x_4 \geq 1 \\
&x_1 + x_2 + x_3 + x_5 \geq 1 \\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 1 \\
&x_1 + x_3 + x_4 + x_6 + x_7 \geq 1 \\
&x_2 + x_3 + x_5 + x_6 \geq 1 \\
&x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \geq 1 \\
&x_4 + x_6 + x_7 + x_8 \geq 1 \\
&x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \geq 1 \\
&x_5 + x_8 + x_9 + x_{10} + x_{11} \geq 1 \\
&x_8 + x_9 + x_{10} + x_{11} \geq 1 \\
&x_9 + x_{10} + x_{11} \geq 1 \\
&x_i \text{ are binary for all } i
\end{aligned}
$$

4. Let x_1 be the number of product A produced each day, and let x_2 be the number of product B produced each day. Then, the following integer program will determine how much of each product to create. (The inequalities basically say that either $x_1 \leq 20$ and $x_2 \leq 10$ or $x_1 \leq 12$ and $x_2 \leq 25$.)

$$
\begin{aligned}
&\text{Max } 10x_1 + 12x_2 \\
&\text{subject to} \\
&x_1 \leq 20 + My \\
&x_1 \leq 12 + M(1-y) \\
&x_2 \leq 10 + My \\
&x_2 \leq 25 + M(1-y) \\
&x_1 + x_2 \leq 35 \\
&x_1, x_2 \geq 0 \\
&x_1, x_2 \text{ are integers} \\
&y \text{ is binary}
\end{aligned}
$$

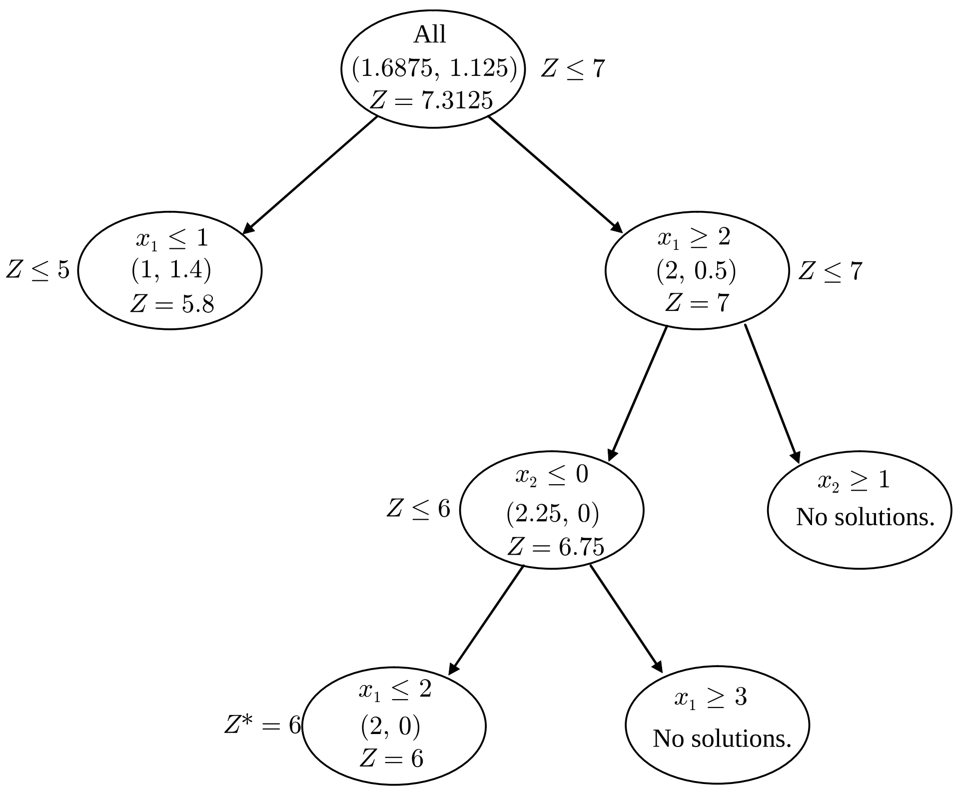
The value of M is a large number. For example, $M = 25$ would work.

5. The resulting tree is:



The solution to the BIP is $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $Z = 2$.

6. The resulting tree is:



The solution to the integer program is $x_1 = 2$, $x_2 = 0$, $Z = 6$.

7. (a) $x \approx 1.3125, f(x) \approx 61.56124$
   (b) $x \approx 1.345382, f(x) \approx 61.66502$

8. (a) There are two critical points: $(0,0)$ and $(0,2)$.
   (b) The point $(0,0)$ is a saddle point; the point $(0,2)$ is a local min.

9. (a) First, we compute the Hessian:

$$\text{Hessian} = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}$$

   The determinant is 4, and the trace is -6. Thus, the product of the eigenvalues is positive and the sum of the eigenvalues is negative. Thus, both eigenvalues are negative, so f is concave.

   (b) The sequence of points is: $(0,0) \to (1.2, -0.4) \to (1.6, 0.8)$.
   Thus, the answer is: $\mathbf{x}^* \approx (1.6, 0.8)$

10. The maximum occurs at $(-6, 0)$. The maximum value is 60.

11. The KKT conditions are:

    1. $2 - 2x_1 - u \leq 0$ and $3 - 2x_2 - u \leq 0$
    2. $x_1(2 - 2x_1 - u) = 0$ and $x_2(3 - 2x_2 - u) = 0$
    3. $x_1 + x_2 - 2 \leq 0$
    4. $u(x_1 + x_2 - 2) = 0$
    5. $x_1, x_2 \geq 0$
    6. $u \geq 0$

    The solution is $x_1 = 3/4$ and $x_2 = 5/4$. (Also, $u = 1/2$.)