An Important Proposition

Most of you proved the following proposition as part of Exercise 1 on Homework 6. But since it wasn't stated explicitly, and since we're using it in the most recent set of notes (in the proof of the integral Hölder's inequality, Theorem 13), I thought I should state and prove it here. You should feel free to use this proposition on Homework 7 and all subsequent homework assignments.

Proposition

Let (X, μ) be a measure space, and let f be a non-negative measurable function on X. If

$$\int_X f \, d\mu = 0$$

then $f = 0$ almost everywhere.

PROOF For each n, let $E_n = f^{-1}\big((1/n, \infty]\big)$. Note then that each E_n is measurable. But

$$0 = \int_X f \, d\mu \geq \int_{E_n} f \, d\mu \geq \int_{E_n} \frac{1}{n} \, d\mu = \frac{\mu(E_n)}{n}$$

for each n, and therefore each $\mu(E_n) = 0$. Then

$$E = \bigcup_{n \in \mathbb{N}} E_n = f^{-1}\big((0, \infty]\big)$$

is a set of measure zero, and $f = 0$ on E^c. ■

Corollary

Let (X, μ) be a measure space, let f be a measurable function on X, and let $p \in [1, \infty)$. Then $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere.

PROOF If $f = 0$ almost everywhere, then $|f|^p = 0$ almost everywhere, and it follows that $\|f\|_p = 0$. Conversely, if $\|f\|_p = 0$, then

$$\int_X |f|^p \, d\mu \;=\; 0.$$

Since $|f|^p \geq 0$, it follows that $|f|^p = 0$ almost everywhere, and hence $f = 0$ almost everywhere. ■