Banach Spaces

These notes provide an introduction to Banach spaces, which are complete normed vector spaces. For the purposes of these notes, all vector spaces are assumed to be over the real numbers.

Complete Metric Spaces

Notions such as convergent sequence and Cauchy sequence make sense for any metric space.

Definition: Cauchy Sequence, Convergent Sequence
Let X be a metric space, and let $\{x_n\}$ be a sequence of points in X.

1. We say that $\{x_n\}$ is a **Cauchy sequence** if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that
$$i, j \geq N \quad \Rightarrow \quad d(x_i, x_j) < \epsilon.$$
2. We say that $\{x_n\}$ **converges** to a point $x \in X$ if
$$\lim_{n\to\infty} d(x_n, x) \,=\, 0.$$

Proposition 1 Convergent Sequences are Cauchy

If X is a normed vector space, then every convergent sequence in X is a Cauchy sequence.

PROOF Let $\{x_n\}$ be a sequence that converges to some point $x \in X$, and let $\epsilon > 0$.

Since $(x_n, x) \to 0$, there exists an $N \in \mathbb{N}$ so that

$$n \geq N \quad \Rightarrow \quad d(x_n, x) < \frac{\epsilon}{2}.$$

If $i, j \geq N$ it follows that

$$d(x_i, x_j) \;\leq\; d(x_i, x) + d(x_j, x) \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon,$$

which proves that $\{x_n\}$ is a Cauchy sequence. ■

Though every convergent sequence is Cauchy, it is not necessarily the case that every Cauchy sequence in a metric space converges. For example, let $\mathbb{Q}$ be the metric space of all rational numbers under the usual metric:

$$d(q_1, q_2) \;=\; |q_1 - q_2|.$$

Then there are many Cauchy sequences in $\mathbb{Q}$ that do not converge to any point in $\mathbb{Q}$. For example, the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \quad \ldots$$

is a Cauchy sequence in $\mathbb{Q}$, but it does not converge to any point in $\mathbb{Q}$.

Definition: Complete Metric Space
A metric space X is said to be **complete** if every Cauchy sequence in X converges to a point in X.

For example, the metric space $\mathbb{R}$ of real numbers is complete, since every Cauchy sequence in $\mathbb{R}$ converges. More generally, $\mathbb{R}^n$ is a complete metric space under the usual metric for all $n \in \mathbb{N}$.

Of course, any normed vector space V is naturally a metric space, with metric defined by

$$d(\mathbf{v}, \mathbf{w}) \;=\; \|\mathbf{v} - \mathbf{w}\|.$$

Thus it makes sense to talk about convergent sequences and Cauchy sequences in a normed metric space. Specifically, if $\{\mathbf{v}_n\}$ a sequence of vectors in V, then $\{\mathbf{v}_n\}$ is Cauchy if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that

$$i, j \geq N \quad \Rightarrow \quad \|\mathbf{v}_i - \mathbf{v}_j\| < \epsilon,$$

and $\{\mathbf{v}_n\}$ converges to a vector $\mathbf{v} \in V$ if $\|\mathbf{v}_n - \mathbf{v}\| \to 0$ as $n \to \infty$.

Definition: Banach Space
A normed vector space V is called a **Banach space** if every Cauchy sequence in V converges.

That is, a Banach space is a complete normed vector space. It is not hard to prove that any finite-dimensional normed vector space is a Banach space (see the appendix at the end of these notes), so completeness is really only an issue for infinite-dimensional spaces.

Infinite-Dimensional Vector Spaces

A vector space V is said to be **infinite-dimensional** if V does not have any finite basis. Perhaps the simplest example of an infinite-dimensional vector space is the space $\mathbb{R}^\omega$ consisting of all infinite sequences of real numbers:

$$\mathbb{R}^\omega = \{(v_1, v_2, v_3, \ldots) \mid v_1, v_2, v_3, \ldots \in \mathbb{R}\}.$$

Unfortunately, there is not an obvious way to put a norm on $\mathbb{R}^\omega$. In particular, the formula

$$\|(v_1, v_2, v_3, \ldots,)\| = \sqrt{\sum_{n=1}^{\infty} v_n^2}$$

does *not* define a norm, since the sum on the right may diverge. For example,

$$\|(1, 1, 1, \ldots)\| = \sqrt{1^2 + 1^2 + 1^2 + \cdots} = \infty.$$

If we want an infinite-dimensional normed space, we must restrict to a *subspace* of $\mathbb{R}^\omega$. The following example discusses one such subspace.

EXAMPLE 1 The Space $\mathbb{R}^\infty$
Consider the follows subset of $\mathbb{R}^\omega$:

$$\mathbb{R}^\infty = \big\{(v_1, v_2, v_3, \ldots) \in \mathbb{R}^\omega \;\big|\; v_n = 0 \text{ for all but finitely many } n\big\}.$$

Note that $\mathbb{R}^\infty$ is closed under addition and scalar multiplication, and is therefore a linear subspace of $\mathbb{R}^\omega$. Moreover, the function

$$\|(v_1, v_2, v_3, \ldots,)\| = \sqrt{\sum_{n=1}^{\infty} v_n^2}$$

is a valid norm on $\mathbb{R}^\infty$, since the sum on the right only ever has finitely many nonzero terms.

Unfortunately, $\mathbb{R}^\infty$ is not a Banach space. For example, let $\{\mathbf{v}_n\}$ be the following sequence of vectors in $\mathbb{R}^\infty$:

$$\mathbf{v}_1 = (1,0,0,0,\ldots), \quad \mathbf{v}_2 = \left(1,\tfrac{1}{2},0,0,0,\ldots\right), \quad \mathbf{v}_3 = \left(1,\tfrac{1}{2},\tfrac{1}{3},0,0,0,\ldots\right), \quad \ldots$$

Let $\mathbf{v} = \left(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right)$. Then

$$\lim_{n\to\infty} \|\mathbf{v}_n - \mathbf{v}\| = \lim_{n\to\infty} \sqrt{\sum_{k=n+1}^{\infty} \frac{1}{k^2}} = 0,$$

so $\mathbf{v}_n \to \mathbf{v}$ in $\mathbb{R}^\omega$. It follows that $\{\mathbf{v}_n\}$ is a Cauchy sequence. However, the vector $\mathbf{v}$ does not lie in $\mathbb{R}^\infty$, so $\{\mathbf{v}_n\}$ is a Cauchy sequence in $\mathbb{R}^\infty$ that does not converge to any point in $\mathbb{R}^\infty$. ■

If we want a Banach space of sequences, we must include at least some sequences with infinitely many nonzero terms.

Theorem 2 ℓ^2 is a Banach Space

The set

$$\ell^2 = \left\{ (v_1, v_2, v_3, \ldots) \in \mathbb{R}^\omega \;\middle|\; \sum_{n=1}^{\infty} v_n^2 < \infty \right\}$$

forms a Banach space under the norm

$$\|(v_1, v_2, v_3, \ldots)\| = \sqrt{\sum_{n=1}^{\infty} v_n^2}.$$

PROOF Since $\ell^2 = L^2(\mathbb{N})$, it is closed under addition and scalar multiplication, and every Cauchy sequence in ℓ^2 converges by the L^p completeness theorem. ■

By the way, it is not hard to show that every point in ℓ^2 is the limit of a sequence of points in $\mathbb{R}^\infty$. Thus $\mathbb{R}^\infty$ sits inside of ℓ^2 in roughly the same way that the rational numbers sit inside the real numbers. In the language of metric spaces, ℓ^2 is the **metric completion** of $\mathbb{R}^\infty$.

Of course, the above theorem can be generalized to any $p \in [1,\infty]$.

Theorem 3 ℓ^p is a Banach Space

For any $p \in [1, \infty]$, the vector space ℓ^p is a Banach space with respect to the p-norm.

Function Spaces

A **function space** is a vector space whose "vectors" are functions. For example, the set $\mathbb{R}^{\mathbb{R}}$ of all functions $\mathbb{R} \to \mathbb{R}$ forms a vector space, with addition and scalar multiplication defined by

$$(f+g)(x) = f(x) + g(x) \qquad \text{and} \qquad (\lambda f)(x) = \lambda\, f(x).$$

Again, there is not an obvious choice for a norm on $\mathbb{R}^{\mathbb{R}}$, essentially because $\mathbb{R}^{\mathbb{R}}$ is too large a space. However, if we restrict our functions a bit we can define normed vector spaces.

EXAMPLE 2 The Space $C([-1,1])$.
Let $C([-1,1])$ denote the space of all *continuous* functions $[-1,1] \to \mathbb{R}$. Note that this is a vector space, since the sum of continuous functions is continuous, and any scalar multiple of a continuous function is continuous. We can define a norm on $C([-1,1])$ by the formula

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx.$$

Note that this integral is always finite, since every continuous function on $[-1,1]$ is bounded.

Unfortunately, $C([-1,1])$ is not a Banach space with respect to this norm. For example let $\{f_n\}$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} -1 & \text{if } x \in \left[-1, -\frac{1}{n}\right], \\ nx & \text{if } x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 1 & \text{if } x \in \left[\frac{1}{n}, 1\right]. \end{cases}$$

Note that each f_n is continuous. Furthermore, it is easy to check that

$$\|f_i - f_j\| = \left|\frac{1}{i} - \frac{1}{j}\right|$$

for all $i, j \in \mathbb{N}$, so $\{f_n\}$ is a Cauchy sequence. However, $\{f_n\}$ does not converge (in L^1) to any continuous function $f\colon [-1,1] \to \mathbb{R}$. ■

The trouble here is that sequences of continuous functions can converge to discontinuous functions, so the space of all continuous functions is not complete.

We can use the Lebesgue integral to define something like a complete function space, but it is a bit tricky. Given a measure space (X, μ), one obvious candidate is the space

$$\mathcal{L}^1(X)$$

of all L^1 functions on X, under the L^1 norm. Unfortunately, this space $\mathcal{L}^1(X)$ is *not* a normed vector space. The trouble is that a normed vector space V must satisfy

$$\|\mathbf{v}\| = 0 \quad \Rightarrow \quad \mathbf{v} = \mathbf{0}$$

for all $\mathbf{v} \in V$, but the space $\mathcal{L}^1(X)$ only satisfies

$$\|f\|_1 = 0 \quad \Rightarrow \quad f = 0 \text{ almost everywhere.}$$

To fix this problem, we must consider a function f to be *equal* to the zero if $f = 0$ almost everywhere. Indeed, we must relax our notion of equality so that f and g are considered equal if $f = g$ almost everywhere.

Definition: L^1-Space
If (X, μ) is a measure space, the corresponding ***L^1*-space**, denoted $L^1(X)$, is the space of all L^1 functions $X \to \mathbb{R}$, where two functions are considered the same if they are equal almost everywhere.

Formally speaking, each element of $L^1(X)$ is an equivalence class of L^1 functions on X, where two functions f and g are considered equivalent if $f = g$ almost everywhere (often abbreviated $f = g$ a.e.). Note that addition and scalar multiplication are well-defined on $L^1(X)$, since

$$f_1 = f_2 \text{ a.e.} \quad \text{and} \quad g_1 = g_2 \text{ a.e.} \quad \Rightarrow \quad f_1 + g_1 = f_2 + g_2 \text{ a.e..}$$

and

$$f_1 = f_2 \text{ a.e.} \quad \Rightarrow \quad \lambda f_1 = \lambda f_2 \text{ a.e.}$$

for all $\lambda \in \mathbb{R}$. From an algebraic point of view, $L^1(X)$ is simply the quotient $\mathcal{L}^1(X)/\mathcal{N}$, where $\mathcal{L}^1(X)$ is the vector space of all L^1 functions on X and $\mathcal{N}$ is the linear subspace of $\mathcal{L}^1(X)$ consisting of all functions that are equal to zero almost everywhere.

Theorem 4 $L^1(X)$ is a Banach space

If (X, μ) is a measure space, then $L^1(X)$ is a Banach space under the L^1 norm.

PROOF This follows immediately from the L^1 completeness theorem. ■

In the case of Lebesgue measure, the space $L^1(X)$ can be viewed as the metric completion of the space of continuous functions.

Theorem 5 Density of Continuous Functions

For any $f \in L^1(\mathbb{R})$, there exists a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ so that $f_n \to f$ in L^1.

PROOF See Homework 7, problem 2. ■

It follows that $L^1([a,b])$ is the metric completion of $C([a,b])$ under the L^1 norm for any closed interval $[a,b] \subseteq \mathbb{R}$. From this point of view, the Riemann integral

$$R(f) = \int_a^b f(x)\,dx$$

can be thought of as a continuous function $R\colon C([a,b]) \to \mathbb{R}$, and the Lebesgue integral

$$L(f) = \int_{[a,b]} f\,d\mu$$

is simply the continuous extension $L\colon L^1([a,b]) \to \mathbb{R}$ of R to all of $L^1([a,b])$.

Of course, we can generalize all of these results from L^1-space to L^p-space.

Definition: L^p-Space
If (X,μ) is a measure space and $p \in [1,\infty]$, the corresponding **L^p-space**, denoted $L^p(X)$, is the space of all L^p functions $X \to \mathbb{R}$, where two functions are considered the same if they are equal almost everywhere.

Theorem 6 $L^p(X)$ is a Banach space

If (X,μ) is a measure space and $p \in [1,\infty]$, then $L^p(X)$ is a Banach space under the L^p norm.

By the way, there is one L^p norm under which the space $C([a,b])$ of continuous functions is complete.

Theorem 7 L^∞ Completeness of $C([a,b])$

For each closed interval $[a,b] \subset \mathbb{R}$, the vector space $C([a,b])$ under the L^∞-norm is a Banach space.

PROOF Let $[a,b] \subset \mathbb{R}$ be a closed interval. Note first that

$$\|f\|_\infty = \max\bigl\{|f(x)| \bigm| x \in [a,b]\bigr\}$$

for any continuous function $f\colon [a,b] \to \mathbb{R}$. In particular, if f and g are continuous, then $\|f-g\|_\infty < \epsilon$ if and only if $|f-g| < \epsilon$.

Let $\{f_n\}$ be a Cauchy sequence in $C([a,b])$ under the L^∞-norm. Then for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that

$$i, j \geq N \quad \Rightarrow \quad |f_i - f_j| < \epsilon.$$

It follows that $\{f_n(x)\}$ is a Cauchy sequence for each $x \in [a,b]$, so $\{f_n\}$ converges pointwise to some function $f\colon [a,b] \to \mathbb{R}$.

We claim that $\{f_n\}$ converges uniformly to f. To prove this, let $\epsilon > 0$, and let $N \in \mathbb{N}$ so that

$$i, j \geq N \quad \Rightarrow \quad |f_i - f_j| < \epsilon.$$

If $n \geq N$ and $x \in [a,b]$, then we know that $|f_n(x) - f_j(x)| < \epsilon$ for all $j \geq N$, and it follows that $|f_n(x) - f(x)| \leq \epsilon$. We conclude that $|f_n - f| \leq \epsilon$ for all $n \geq N$, which proves that $f_n \to f$ uniformly. Then f must be continuous by the uniform limit theorem, and $f_n \to f$ in L^∞. ■

Sums in Banach Spaces

A norm on a vector space makes it possible to define infinite series of vectors.

Definition: Infinite Series of Vectors
Let V be a normed vector space, and let $\{\mathbf{v}_n\}$ be a sequence in V. We say that the series

$$\sum_{n=1}^{\infty} \mathbf{v}_n$$

converges in V if the sequence $\mathbf{s}_n = \sum_{k=1}^{n} \mathbf{v}_k$ of partial sums converges to some point $\mathbf{s} \in V$. In this case, $\mathbf{s}$ is called the **sum of the series**.

There is a nice test for convergence of series in a Banach space.

Theorem 8 Absolute Convergence Test

Let V be a Banach space, let $\{\mathbf{v}_n\}$ be a sequence of vectors in V, and suppose that

$$\sum_{n=1}^{\infty} \|\mathbf{v}_n\| < \infty.$$

Then the series

$$\sum_{n=1}^{\infty} \mathbf{v}_n$$

converges in V.

PROOF For each $n \in \mathbb{N}$, let

$$a_n = \sum_{k=1}^{n} \|\mathbf{v}_k\| \qquad \text{and} \qquad \mathbf{s}_n = \sum_{k=1}^{n} \mathbf{v}_k.$$

By hypothesis, the sequence $\{a_n\}$ converges, so it is a Cauchy sequence. In particular, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that

$$i, j \geq n \quad \Rightarrow \quad |a_i - a_j| < \epsilon.$$

But it follows easily from the triangle inequality that $\|\mathbf{s}_i - \mathbf{s}_j\| \leq |a_i - a_j|$ for all $i, j \in \mathbb{N}$, so $\{\mathbf{s}_n\}$ is a Cauchy sequence as well. Since V is a Banach space, we conclude that $\{\mathbf{s}_n\}$ converges in V, so the given series converges. ■

Appendix: Finite-Dimensional Spaces

In this appendix we prove that any finite-dimensional normed vector space is complete. We begin by examining the relationship between different norms on the same vector space.

Definition: Lipschitz Equivalence
Let V be a vector space, and let $\|-\|_A$ and $\|-\|_B$ be two norms on V. We say that $\|-\|_A$ and $\|-\|_B$ are **Lipschitz equivalent** if there exist constants $\lambda, \mu > 0$ so that

$$\|\mathbf{v}\|_A \leq \lambda \|\mathbf{v}\|_B \qquad \text{and} \qquad \|\mathbf{v}\|_B \leq \mu \|\mathbf{v}\|_A$$

for all $\mathbf{v} \in V$.

Note that this is an equivalence relation for norms on V.

Proposition 9 Properties of Lipschitz Equivalence

Let V be a vector space, and let $\|-\|_A$ and $\|-\|_B$ be two Lipschitz equivalent norms on V. Then

1. *A sequence $\{\mathbf{v}_n\}$ is Cauchy with respect to $\|-\|_A$ if and only if it is Cauchy with respect to $\|-\|_B$.*
2. *A sequence $\{\mathbf{v}_n\}$ converges to a vector $\mathbf{v}$ with respect to $\|-\|_A$ if and only if it converges to $\mathbf{v}$ with respect to $\|-\|_B$.*
3. *V is complete with respect to $\|-\|_A$ if and only if it is complete with respect to $\|-\|_B$.*

PROOF Let $\lambda, \mu > 0$ be the required constants. For (1), let $\{\mathbf{v}_n\}$ be a Cauchy sequence with respect to $\|-\|_A$, and let $\epsilon > 0$. There there exists an $N \in \mathbb{N}$ so that

$$n \geq N \quad \Rightarrow \quad \|\mathbf{v}_i - \mathbf{v}_j\|_A < \frac{\epsilon}{\mu}.$$

Since $\|\mathbf{v}_i - \mathbf{v}_j\|_B \leq \mu \|\mathbf{v}_i - \mathbf{v}_j\|_A$ for all i and j, it follows that

$$n \geq N \quad \Rightarrow \quad \|\mathbf{v}_i - \mathbf{v}_j\|_A < \epsilon,$$

so $\{\mathbf{v}_n\}$ is a Cauchy sequence with respect to $\|-\|_B$ as well.

For (2), suppose that $\mathbf{v}_n \to \mathbf{v}$ with respect to $\|-\|_A$, so $\|\mathbf{v}_n - \mathbf{v}\|_A \to 0$ as $n \to \infty$. Since

$$0 \;\leq\; \|\mathbf{v}_n - \mathbf{v}\|_B \;\leq\; \mu \|\mathbf{v}_n - \mathbf{v}\|_A$$

for all n, it follows that $\|\mathbf{v}_n - \mathbf{v}\|_B \to 0$ as $n \to \infty$, so $\mathbf{v}_n \to \mathbf{v}$ with respect to $\|-\|_B$. Statement (3) follows immediately from (1) and (2). ■

It is also possible to prove that Lipschitz equivalent norms on a vector space V define the same topology on V.

Proposition 10 Norms on $\mathbb{R}^n$

> *Every norm on $\mathbb{R}^n$ is Lipschitz equivalent to the Euclidean norm*

PROOF Let $\|-\|_A$ be a norm on $\mathbb{R}^n$, and let $\|-\|_2$ denote the Euclidean norm. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis for $\mathbb{R}^n$, and set

$$\lambda = \sqrt{\|\mathbf{e}_1\|_A^2 + \cdots + \|\mathbf{e}_n\|_A^2}$$

Then for any vector $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, we have

$$\|\mathbf{v}\|_A = \|v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n\|_A \leq |v_1|\,\|\mathbf{e}_1\|_A + \cdots + |v_n|\,\|\mathbf{e}_n\|_A.$$

so by the Cauchy-Schwarz inequality

$$\|\mathbf{v}\|_A \leq \sqrt{|v_1|^2 + \cdots + |v_n|^2}\sqrt{\|e_1\|_A^2 + \cdots + \|e_n\|_A^2} = \lambda\,\|\mathbf{v}\|_2,$$

which is half of what we need to prove.

For the other half, observe that

$$\big|\|\mathbf{v}\|_A - \|\mathbf{w}\|_A\big| \leq \|\mathbf{v} - \mathbf{w}\|_A \leq \lambda\|\mathbf{v} - \mathbf{w}\|_2$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and therefore $\|-\|_A$ is a continuous function on $\mathbb{R}^n$. Consider the unit sphere

$$S^{n-1} = \big\{\mathbf{u} \in \mathbb{R}^n \;\big|\; \|\mathbf{u}\|_2 = 1\big\}.$$

This set is closed and bounded in $\mathbb{R}^n$, and is therefore compact. Then $\|-\|_A$ attains a minimum value α on S^{n-1}, which must be greater than zero. Thus

$$\|\mathbf{u}\|_A \geq \alpha$$

for all $\mathbf{u} \in S^{n-1}$. It follows easily that

$$\|\mathbf{v}\|_A \geq \alpha\,\|\mathbf{v}\|_2$$

for all $\mathbf{v} \in \mathbb{R}^n$. Equivalently,

$$\|\mathbf{v}\|_2 \leq \mu\,\|\mathbf{v}\|_A$$

for all $\mathbf{v} \in \mathbb{R}^n$, where $\mu = 1/\alpha$. ■

An easily corollary of this is that $\mathbb{R}^n$ is complete with respect to any norm. Indeed, we can extend this to any finite-dimensional vector space.

Corollary 11 Completeness of Finite-Dimensional Spaces

Every finite-dimensional normed vector space is a Banach space.

PROOF Let V be a normed vector space of dimension n, where $\|-\|_V$ is the norm on V, and let $T\colon \mathbb{R}^n \to V$ be a linear isomorphism. Define a norm $\|-\|_T$ on $\mathbb{R}^n$ by

$$\|\mathbf{w}\|_T \;=\; \|T(\mathbf{w})\|_V.$$

Then $\|-\|_T$ is Lipschitz equivalent to the Euclidean norm, and hence $\mathbb{R}^n$ is complete with respect to $\|-\|_T$. But T is an isometric isomorphism with respect to $\|-\|_T$, and hence V is complete as well. ■

Exercises

1. (a) Let X be a metric space, and let $\{x_n\}$ be a Cauchy sequence in X. Prove that if $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ converges.
   (b) Deduce that every compact metric space is complete.

2. Let X be a complete metric space, let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ be a sequence of closed sets in X, and let

   $$\operatorname{diam}(F_n) \;=\; \sup\{d(x,y) \mid x, y \in F_n\}.$$

   Prove that if $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$, then $\bigcap_{n\in\mathbb{N}} F_n$ is nonempty.

3. Let X be a complete metric space, let F be a closed subset of X, and let $d|_F$ be the metric on F obtained by restricting the metric on X. Prove that F is a complete metric space with respect to $d|_F$.

4. A sequence $\{x_n\}$ in a metric space X is said to have **bounded variation** if

   $$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \;<\; \infty.$$

   Prove that if X is complete, then every sequence of bounded variation in X converges.

5. Prove that $\mathbb{R}^\infty$ is dense in ℓ^2. That is, prove that every point in ℓ^2 is the limit of a sequence of points in $\mathbb{R}^\infty$.

6. If (X,μ) is a measure space, prove that the simple functions are dense in $L^1(X)$.

7. Let (X, μ) be a measure space, and define a function $L\colon L^1(X) \to \mathbb{R}$ by

$$L(f) = \int_X f\, d\mu.$$

Prove that L is continuous.

8. Prove that if $\{a_n\}$ is an ℓ^2 sequence, then the Taylor series

$$\sum_{n=1}^{\infty} a_n x^n$$

converges in $L^1([-1,1])$.

9. Let $p \in [1,\infty]$. Prove that if $\{a_n\}$ is an ℓ^1 sequence, then the Fourier series

$$\sum_{n=1}^{\infty} a_n \cos nx$$

converges in $L^1([-\pi,\pi])$.

10. Let $p \in (0,1)$, and let $\mathbf{a} = (a_1, a_2, \ldots)$ be a point in ℓ^p. Prove that

$$\mathbf{a} = \sum_{n=1}^{\infty} a_n \mathbf{e}_n.$$

in ℓ^p, where $\{\mathbf{e}_n\}$ is the sequence

$$\mathbf{e}_1 = (1,0,0,0,\ldots), \quad \mathbf{e}_2 = (0,1,0,0,\ldots), \quad \ldots$$

11. Let V be a vector space. Prove that Lipschitz equivalence is an equivalence relation for norms on V.

12. If V is a vector space, prove that any two Lipschitz equivalent norms induce the same topology on V.

13. Let $[a,b]$ be a closed interval, let $g\colon [a,b] \to (0,\infty)$ be a continuous function, and let $d\nu = g\, dm$, where m denotes Lebesgue measure. Prove that the the norm $\|{-}\|_{1,\nu}$ on $L^1([a,b])$ defined by

$$\|f\|_{1,\nu} = \int_{[a,b]} |f|\, d\nu$$

is Lipschitz equivalent to the usual 1-norm.