The Lebesgue Integral

Having completed our study of Lebesgue measure, we are now ready to consider the Lebesgue integral. Before diving into the details of its construction, though, we would like to give a broad overview of the subject. Most of the propositions and theorems in these notes will not have proofs, though all of this will be proven later.

Given a measure space $(X, \mathcal{M}, \mu)$ and a function $f\colon X \to \mathbb{R}$, we wish to define the **Lebesgue integral** of f on X, denoted

$$\int_X f \, d\mu.$$

Of course, it is too much to hope that every function $f\colon X \to \mathbb{R}$ would be integrable, and some restriction on f will be necessary.

Measurable Functions

To get a sense of the kinds of functions we will need to exclude, recall that the **characteristic function** (or **indicator function**) of a set $S \subseteq X$ is the function $\chi_S\colon X \to \mathbb{R}$ defined by

$$\chi_S(x) \;=\; \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

If $E \subseteq X$ is a measurable set, it seems intuitive that the integral of χ_E should be the same as the measure of E, i.e.

$$\int_X \chi_E \, d\mu \;=\; \mu(E).$$

However, if $S \subseteq X$ is a non-measurable set, then the same intuition suggests that we should not be able to assign a value to the integral of χ_S. For example, if $V \subseteq [0, 1]$

is a Vitali set, then the integral

$$\int_{\mathbb{R}} \chi_V \, dm$$

should be undefined.

The following definition describes those functions that ought to be "reasonable" enough to integrate.

Definition: Measurable Function
Let $(X, \mathcal{M}, \mu)$ be a measure space. A function $f\colon X \to [-\infty, \infty]$ is said to be **measurable** if the set

$$f^{-1}\big((a,\infty]\big) \;=\; \{x \in X \mid f(x) > a\}$$

is measurable for each $a \in \mathbb{R}$.

For example, if $\chi_V : \mathbb{R} \to \mathbb{R}$ is the characteristic function of a Vitali set, then

$$\chi_V^{-1}\big((\tfrac{1}{2},\infty]\big) \;=\; V$$

is not a measurable set, and hence χ_V is not a measurable function.

Trouble with Negative Values

Even if we restrict to measurable functions, it is not reasonable to expect to be able to evaluate any integral. For example, consider the integral

$$\int_{\mathbb{R}} \cos \, dm.$$

Though the cosine function is measurable, each full period of cosine bounds an area of 2 above the x-axis and an area of 2 below the x-axis, which means that this integral corresponds to the infinite sum

$$\cdots + (-2) + 2 + (-2) + 2 + (-2) + 2 + (-2) + \cdots$$

Since there is no reasonable way to assign a value to this sum, the Lebesgue integral of the cosine function on all of $\mathbb{R}$ is undefined.

In general, if $(X, \mathcal{M}, \mu)$ is a measure space, then we will be able to define the Lebesgue integral

$$\int_X f \, d\mu$$

for any *non-negative* measurable function $f\colon X \to [0,\infty]$. Depending on the function f, this integral may be infinite, but it will always have a well-defined value in $[0,\infty]$. For the purposes of these notes, we assume that the Lebesgue integral can be defined in this case.

Assumption: Lebesgue Integral for Non-Negative Functions
Let $(X,\mathcal{M},\mu)$ be a measure space, and let $f\colon X \to [0,\infty]$ be a non-negative measurable function. Then the Lebesgue integral

$$\int_X f\,d\mu$$

has a well-defined value in $[0,\infty]$.

For measurable functions $f\colon X \to [-\infty,\infty]$ with both positive and negative values, we must analyze the positive and negative parts separately.

Definition: Positive and Negative Parts of a Function
Let X be a set, and let $f\colon X \to [-\infty,\infty]$. Then the **positive part** and **negative part** of f are the functions $f^+, f^-\colon X \to [0,\infty]$ defined by

$$f^+(x) \;=\; \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f^-(x) \;=\; \begin{cases} -f(x) & \text{if } f(x) \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$f \;=\; f^+ - f^-$$

for any function $f\colon X \to [-\infty,\infty]$.

Proposition 1

Let $(X,\mathcal{M},\mu)$ be a measure space, and let $f\colon X \to [-\infty,\infty]$ be a measurable function. Then the positive and negative parts of f are measurable.

PROOF If $a \in \mathbb{R}$, then clearly $(f^+)^{-1}\big((a,\infty]\big) = X$ for $a < 0$, and

$$(f^+)^{-1}\big((a,\infty]\big) \;=\; f^{-1}\big((a,\infty]\big)$$

for $a \geq 0$, and so f^+ is measurable. For f^-, we similarly have $(f^-)^{-1}\big((a,\infty]\big) = X$

for $a < 0$. For $a \geq 0$, observe that

$$(f^-)^{-1}\big((a,\infty]\big) \;=\; f^{-1}\big([-\infty,-a)\big).$$

But the set on the right is the complement of $f^{-1}\big([-a,\infty]\big)$, and

$$f^{-1}\big([-a,\infty]\big) \;=\; \bigcap_{n\in\mathbb{N}} f^{-1}\big((-a-\tfrac{1}{n},\infty]\big),$$

which is measurable. ■

This allows us to define the Lebesgue integral for measurable functions that take both positive and negative values.

Definition: Lebesgue Integral of Signed Functions
Let $(X,\mathcal{M},\mu)$ be a measure space, and let $f\colon X \to [-\infty,\infty]$ be a measurable function. We say that f is **Lebesgue integrable** if either

$$\int_X f^+\,d\mu \;<\; \infty \qquad \text{or} \qquad \int_X f^-\,d\mu \;<\; \infty.$$

In this case, the **Lebesgue integral** of f on X is defined by

$$\int_X f\,d\mu \;=\; \int_X f^+\,d\mu \;-\; \int_X f^-\,d\mu.$$

Incidentally, a measurable function $f\colon X \to \mathbb{R}$ is said to have **type $\boldsymbol{L^1}$** if *both* of the integrals

$$\int_X f^+\,d\mu \qquad \text{and} \qquad \int_X f^-\,d\mu$$

are finite. Equivalently, f has type L^1 if

$$\int_X |f|\,d\mu \;<\; \infty.$$

Every L^1 function is Lebesgue integrable, but a Lebesgue integrable function whose integral is either ∞ or $-\infty$ is not L^1.

Elementary Properties

The following proposition lists many elementary properties of measurable functions and the Lebesgue integral that we will prove in time. For the third property, two measurable functions f and g on a measure space X are said to be **equal almost everywhere** if they are equal on the complement of a set of measure zero. That is, $f = g$ almost everywhere if there exists a set $Z \subseteq X$ of measure zero such that $f(x) = g(x)$ for all $x \in X - Z$.

Proposition 2 Properties of the Lebesgue Integral

Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. *If $E \subseteq X$ is measurable, then χ_E is a measurable function, and*

$$\int_X \chi_E \, d\mu \;=\; \mu(E).$$

2. *If f and g are Lebesgue integrable functions on X and $f \leq g$, then*

$$\int_X f \, d\mu \;\leq\; \int_X g \, d\mu.$$

3. *If f, g are measurable function on X and $f = g$ almost everywhere, then f is Lebesgue integrable if and only if g is Lebesgue integrable, in which case*

$$\int_X f \, d\mu \;=\; \int_X g \, d\mu.$$

4. *If f is a measurable function on X, then so is $|f|$, and*

$$\left| \int_X f \, d\mu \right| \;\leq\; \int_X |f| \, d\mu.$$

5. *If $f \colon X \to [-\infty, \infty]$ is a measurable function and k is a finite constant, then kf is measurable. Moreover, if f is Lebesgue integrable, then so is kf, with*

$$\int_X kf \, d\mu \;=\; k \int_X f \, d\mu.$$

6. *If $f, g \colon X \to [-\infty, \infty]$ are measurable functions and $(f+g)(x)$ is defined for all $x \in X$, then $f + g$ is measurable. Moreover,*

$$\int_X (f+g) \, d\mu \;=\; \int_X f \, d\mu + \int_X g \, d\mu$$

whenever the sum of integrals on the right is defined.

Integration on Subsets

Sometimes we want to integrate a function on just part of a measure space. For example, given a measurable function $f\colon \mathbb{R} \to [-\infty,\infty]$, we might want to integrate f on a closed interval $[a,b]$:

$$\int_{[a,b]} f\,dm.$$

More generally, if $(X,\mathcal{M},\mu)$ is any measure space and E is a measurable subset of X, we want to define the integral

$$\int_E f\,d\mu.$$

The following definition allows for this sort of integral.

Definition: Integral on a Subset
Let $(X,\mathcal{M},\mu)$ be a measure space, let f be a measurable function on X, and let E be a measurable subset of X. Then the **Lebesgue integral of f on E** is defined as follows:

$$\int_E f\,d\mu \;=\; \int_X f\,\chi_E\,d\mu.$$

Note that the function $f\,\chi_E$ is indeed measurable, since

$$(f\,\chi_E)^{-1}\big((a,\infty]\big) \;=\; E \cap f^{-1}\big((a,\infty]\big)$$

for any $a \geq 0$, and

$$(f\,\chi_E)^{-1}\big((a,\infty]\big) \;=\; E^c \cup \Big(E \cap f^{-1}\big((a,\infty]\big)\Big)$$

for any $a < 0$. Of course, it's possible that $f\,\chi_E$ isn't Lebesgue integrable, in which case the integral

$$\int_E f\,d\mu$$

is undefined. In general, we say that f is **Lebesgue integrable on E** if the function $f\,\chi_E$ is Lebesgue integrable on X.

Of course, this definition isn't really very satisfying. For one thing, we shouldn't need f to be defined on all of X to be able to integrate it on E. If we really want a satisfying definition of integration on E, we need to make E itself into a measure space.

Definition: Restrictions of Measures
Let $(X, \mathcal{M}, \mu)$ be a measure space, and let E be a measurable subset of X.

1. The **restriction of $\mathcal{M}$ to E**, denoted $\mathcal{M}|_E$, is the collection of all subsets of E that lie in $\mathcal{M}$.

2. The **restriction of μ to E**, denoted $\mu|_E$ is the restriction of the measure μ to the collection $\mathcal{M}|_E$.

It is easy to check that $\mathcal{M}|_E$ is a σ-algebra on E and $\mu|_E$ is a measure on E, and hence $(E, \mathcal{M}|_E, \mu|_E)$ is a measure space. The following proposition states that integration with respect to this measure is the same as our previous definition.

Proposition 3 Integrals on Subsets

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a measurable function on X. Then for any measurable set $E \subseteq X$, the restriction $f|_E \colon E \to [-\infty, \infty]$ is a measurable function on E, and

$$\int_E f|_E \, d\mu|_E \;=\; \int_E f \, d\mu.$$

In general, if f is a measurable function whose domain includes E, we will always write

$$\int_E f \, d\mu$$

for the integral

$$\int_E f|_E \, d\mu|_E.$$

Relation to Riemann Integrals

If $[a, b]$ is a closed interval and $f \colon [a, b] \to \mathbb{R}$ is a continuous function, we now have two ways of integrating f, namely the Riemann integral

$$\int_a^b f(x) \, dx$$

and the Lebesgue integral

$$\int_{[a,b]} f \, dm$$

where m denotes Lebesgue measure on $\mathbb{R}$. The following proposition states that these two integrals are in fact the same.

Proposition 4 Riemann vs. Lebesgue Integrals

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on a closed interval. Then f is measurable, and

$$\int_{[a,b]} f\, dm \;=\; \int_a^b f(x)\, dx.$$

Incidentally, an *improper* Riemann integral is not always the same thing as a Lebesgue integral. For example, suppose that $f\colon [1,\infty) \to \mathbb{R}$ is a continuous function with zeroes at the natural numbers which is positive on $(n, n+1)$ for n odd, negative on $(n, n+1)$ for n even, and satisfies

$$\int_n^{n+1} f(x)\, dx \;=\; \frac{(-1)^{n+1}}{n}$$

for all $n \in \mathbb{N}$. Then the improper Riemann integral of f exists, with

$$\int_1^{\infty} f(x)\, dx \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \;=\; \log 2,$$

but the Lebesgue integral

$$\int_{[1,\infty)} f\, dm$$

is undefined, since f has infinite area both above and below the x-axis. The trouble here is that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally, but our definition of the Lebesgue integral really requires the area to converge absolutely.

Convergence Theorems

As we have stated previously, our goal in developing Lebesgue theory is to have a theory of integration that works well with limits. Ideally, we would like to prove that

$$\lim_{n\to\infty} \int_X f_n\, d\mu \;=\; \int_X \lim_{n\to\infty} f_n\, d\mu$$

for any sequence $\{f_n\}$ of integrable functions that converges pointwise.

However, as the following two examples show, there are certain very fundamental obstacles to switching limits and integrals.

EXAMPLE 1 Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be the sequence of characteristic functions of the intervals $[n, n+1]$, i.e.

$$f_n(x) \;=\; \begin{cases} 1 & \text{if } x \in [n, n+1], \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ pointwise, since $f_n(x) = 0$ for all $n \geq x$, but

$$\lim_{n\to\infty} \int_X f_n \, dx \;=\; \lim_{n\to\infty} m\big([n, n+1]\big) \;=\; 1.$$

Geometrically, the area under the graph of f_n is a rectangle on the plane that moves right to infinity. Note that this example could easily be modified to make the functions f_n continuous, e.g. by replacing the rectangle by a bump of constant area. ■

EXAMPLE 2 Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) \;=\; \begin{cases} n & \text{if } x \in (0, 1/n), \\ 0 & \text{otherwise.} \end{cases}$$

Then again $f_n \to 0$ pointwise, since $f_n(x) = 0$ as long as $1/n < x$, but

$$\int_X f_n \, d\mu \;=\; 1$$

for each n, so

$$\lim_{n\to\infty} \int_X f_n \, dx \;=\; 1.$$

Geometrically, the area under the graph of f_n is a rectangle of decreasing width and increasing height, with constant area. Note again that this example could be modified to make the functions f_n continuous, e.g. by replacing each rectangle by a triangle of width $1/n$ and height $2n$. ■

As these examples show, there will need to be some nontrivial hypotheses on the sequence $\{f_n\}$ of functions. This leads us to the three classical **convergence theorems**, all of which use a different combination of hypotheses to exclude the examples above.

Before stating the convergence theorems, we state and prove an important property of measurable functions.

Theorem 5 Pointwise Limits of Measurable Functions

Let $(X, \mathcal{M}, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that $\{f_n\}$ converges pointwise to a function $f\colon X \to [-\infty, \infty]$. Then f is measurable.

PROOF Let $a \in \mathbb{R}$. If $x \in X$, observe that $f(x) > a$ if and only if there exists a k so that $f_n(x) > a + 1/k$ for all sufficiently large n. Thus

$$f^{-1}\big((a,\infty]\big) \;=\; \bigcup_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} f_n^{-1}\big((a+\tfrac{1}{k},\infty]\big).$$

Since the functions f_n are measurable, each of the sets $f_n^{-1}\big((a+\frac{1}{k},\infty]\big)$ on the right is measurable, and hence $f^{-1}\big((a,\infty]\big)$ is measurable. ■

Theorem 6 Lebesgue's Monotone Convergence Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let

$$0 \;\leq\; f_1 \;\leq\; f_2 \;\leq\; f_3 \;\leq\; \cdots$$

be a sequence of measurable functions on X. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu \;=\; \int_X \lim_{n\to\infty} f_n \, d\mu.$$

Note that the limits in the monotone convergence theorem are actually supremums, i.e.

$$\sup_{n\in\mathbb{N}} \int_X f_n \, d\mu \;=\; \int_X \sup_{n\in\mathbb{N}} f_n \, d\mu.$$

Theorem 7 Lebesgue's Bounded Convergence Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $\{f_n\}$ be a uniformly bounded, pointwise convergent sequence of measurable functions on X. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu \;=\; \int_X \lim_{n\to\infty} f_n \, d\mu.$$

Here **uniformly bounded** means that there exists a single constant $M > 0$ so that $|f_n| \leq M$ for all n. Note the the philosophy here is quite different from that of the monotone convergence theorem. Instead of requiring the sequence of areas to be nested, we are simply requiring that all of the area is contained in some finite rectangle.

The third convergence theorem is the crown jewel of Lebesgue theory.

Theorem 8 Lebesgue's Dominated Convergence Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a pointwise convergent sequence of measurable functions on X. Suppose that there exists a measurable function $g\colon X \to [0, \infty]$ with

$$\int_X g\, d\mu \;<\; \infty$$

such that $|f_n| \leq g$ for all n. Then

$$\lim_{n\to\infty} \int_X f_n\, d\mu \;=\; \int_X \lim_{n\to\infty} f_n\, d\mu.$$

The function g in the above theorem is known as the **dominating function**. The idea here is that the area between the graphs of $-g$ and g is finite, so there's no way for the area under the functions f_n to escape to infinity if they are confined to this space.

Note that the bounded convergence theorem actually a special case of the dominated convergence theorem, where the dominating function is simply a constant function M.

Exercises

For the following exercises, let $(X, \mathcal{M}, \mu)$ be a measure space.

1. Prove that if f is a measurable function on X, then the set

$$f^{-1}(\infty) \;=\; \{x \in X \mid f(x) = \infty\}$$

is measurable.

2. Let f and g be measurable functions on X, and suppose that $f+g$ is everywhere defined. Prove directly from the definition that $f+g$ is measurable.

3. Let $f\colon X \to [-\infty, \infty]$ be a measurable function. Prove directly from the definition that $-f$ is measurable.

4. Prove that if $S \subseteq X$, then χ_S is a measurable function if and only if S is a measurable set.

5. Let f and g be measurable functions on X, let $E \subseteq X$ be a measurable set, and define a function $h\colon X \to [-\infty, \infty]$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in E, \\ g(x) & \text{if } x \in E^c. \end{cases}$$

Prove that h is measurable.

6. Let f be a Lebesgue integrable function on X. Use the positive and negative parts of f to prove that

$$\left| \int_X f\, d\mu \right| \;\leq\; \int_X |f|\, d\mu.$$

7. Let f be a non-negative measurable function on X, and suppose that $f \leq M$ for some constant M. Prove that

$$\int_E f\, d\mu \;\leq\; M\, \mu(E)$$

for any measurable set $E \subseteq X$.

8. Prove that if $f\colon X \to [-\infty, \infty]$ is Lebesgue integrable on X, then $f\,\chi_E$ is Lebesgue integrable for every measurable set $E \subset X$, and hence all of the integrals

$$\int_E f\, d\mu$$

are defined.

9. Prove that "$f = g$ almost everywhere" is an equivalence relation for measurable functions on X.

10. Let $f\colon X \to [-\infty, \infty]$ be a Lebesgue integrable function, and let $E, F \subseteq X$ be disjoint measurable sets. Prove that

$$\int_{E \cup F} f\, d\mu \;=\; \int_E f\, d\mu + \int_F f\, d\mu.$$

11. Let $\{f_n\}$ be a sequence of non-negative measurable functions on X. Prove that $\sum_{n\in\mathbb{N}} f_n$ is measurable, and that

$$\int_X \sum_{n\in\mathbb{N}} f_n \, d\mu \;=\; \sum_{n\in\mathbb{N}} \int_X f_n \, d\mu.$$

12. Let $f\colon X \to [0,\infty]$ be a measurable function, let $\{E_n\}$ be a sequence of pairwise disjoint, measurable subsets of X, and let $E = \biguplus_{n\in\mathbb{N}} E_n$. Prove that

$$\int_E f \, d\mu \;=\; \sum_{n\in\mathbb{N}} \int_{E_n} f \, d\mu.$$

13. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 x^n \, dx \;=\; 0.$

14. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 \tan^{-1}(nx) \, dx \;=\; \frac{\pi}{2}.$