Non-Measurable Sets

In these notes we will consider the algebraic structure of $\mathbb{R}$ with respect to the rational numbers $\mathbb{Q}$, which has very little to do with the usual geometric and topological structures on $\mathbb{R}$. Using this structure, we will be able to prove some very counterintuitive results about $\mathbb{R}$, including the existence of non-measurable subsets.

Cosets of $\mathbb{Q}$

Let $\mathbb{Q}$ denote the set of rational numbers. A **coset of $\mathbb{Q}$** in $\mathbb{R}$ is any set of the form

$$x + \mathbb{Q} \;=\; \{x + q \mid q \in \mathbb{Q}\}$$

where $x \in \mathbb{R}$. It is easy to see that the cosets of $\mathbb{Q}$ form a partition of $\mathbb{R}$. In particular:

1. If $x, y \in \mathbb{R}$ and $y - x \in \mathbb{Q}$, then $x + \mathbb{Q} = y + \mathbb{Q}$.
2. If $x, y \in \mathbb{R}$ and $y - x \notin \mathbb{Q}$ then $x + \mathbb{Q}$ and $y + \mathbb{Q}$ are disjoint.

Note also that each coset $x+\mathbb{Q}$ is **dense** in $\mathbb{R}$, meaning that every open interval (a, b) in $\mathbb{R}$ contains a point from $x + \mathbb{Q}$.

The collection of all the cosets of $\mathbb{Q}$ in $\mathbb{R}$ is usually[1] denoted $\mathbb{R}/\mathbb{Q}$. Note that there exists a surjection $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ defined by

$$p(x) \;=\; x + \mathbb{Q}$$

for all $x \in \mathbb{R}$. This function p is known as the **canonical surjection**.

Cosets of $\mathbb{Q}$ are interesting because the corresponding partition of $\mathbb{R}$ is almost entirely divorced from the geometry and topology of the real line. Using these cosets, we can create many other structures on $\mathbb{R}$ that violate our geometric intuition. As a simple example of this technique, we give a quick proof of the following proposition.

[1]Those familiar with group theory will recognize $\mathbb{R}/\mathbb{Q}$ as an example of a quotient group, but we will have no need for the group structure on $\mathbb{R}/\mathbb{Q}$ here.

Proposition 1

There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that the image of every open interval (a,b) is all of $\mathbb{R}$.

PROOF Note first that

$$|\mathbb{R}| \;=\; \bigl|\mathbb{Q} \times (\mathbb{R}/\mathbb{Q})\bigr| \;=\; |\mathbb{R}/\mathbb{Q}|$$

The first bijection should be obvious, while the second is an instance of the well-known fact that $|C \times S| = |S|$ for any countable set C and any infinite set S.

Thus there exists a bijection $g\colon \mathbb{R}/\mathbb{Q} \to \mathbb{R}$. Let $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ be the canonical surjection, and let $f = g \circ p$. Then for every $y \in \mathbb{R}$, the preimage $f^{-1}(y)$ is a coset of $\mathbb{Q}$ in $\mathbb{R}$, so every open interval (a,b) contains a point in $f^{-1}(y)$. ■

Note that the graph

$$\Gamma(f) \;=\; \bigl\{\bigl(x, f(x)\bigr) \;\big|\; x \in \mathbb{R}\bigr\}$$

of the function f constructed in the last example is dense in $\mathbb{R}^2$, in the sense that every open disk in $\mathbb{R}^2$ contains a point of the graph. Geometrically, this means that the graph is just a "fog" that fills the plane. The intersection of this fog with each vertical line is a single point (since it is the graph of a function), and the intersection of this fog with each horizontal line is dense on the line.

A Non-Measurable Set

We can use the cosets of $\mathbb{Q}$ in $\mathbb{R}$ described in the last section to construct a subset of $\mathbb{R}$ that is not Lebesgue measurable. The example we give here was first described by Giuseppe Vitali in 1905.

Definition: Vitali Set
A subset $V \subseteq [0,1]$ is called a **Vitali set** if V contains a single point from each coset of $\mathbb{Q}$ in $\mathbb{R}$.

It is easy to construct a Vitali set using the axiom of choice, simply by choosing one element of $(x+\mathbb{Q}) \cap [0,1]$ for each coset $x+\mathbb{Q} \in \mathbb{R}/\mathbb{Q}$. Of course this "construction" is difficult to describe algorithmically, since we are making uncountably many arbitrary choices. Indeed, the axiom of choice is *required* for the construction of a Vitali set, as we will discuss below.

We now turn to the proof that Vitali sets are non-measurable. Given any $S \subseteq \mathbb{R}$ and $t \in \mathbb{R}$ let
$$t + S = \{t + s \mid s \in S\}.$$
That is, $t + S$ is the **translation** of S obtained shifting every point t units to the right on the real line. It is easy to prove that
$$m^*(t + S) = m^*(S)$$
for all $S \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. It follows that $t + E$ is measurable for every measurable set $E \subseteq \mathbb{R}$.

Our goal is to prove the following theorem.

Theorem 2 A Non-Measurable Set

If $V \subseteq [0,1]$ is a Vitali set, then V is not Lebesgue measurable.

We begin with a couple of lemmas.

Lemma 3

Let $V \subseteq [0,1]$ be a Vitali set. Then the sets
$$\{q + V \mid q \in \mathbb{Q}\}$$
are pairwise disjoint, and
$$\mathbb{R} = \biguplus_{q \in \mathbb{Q}} (q + V).$$

PROOF Suppose first that $x \in (q+V) \cap (q'+V)$ for some $q, q' \in \mathbb{Q}$. Then $x = q + v$ and $x = q' + v'$ for some $v, v' \in V$. Then $v = x + (-q)$ and $v' = x + (-q')$, so v and v' both lie in $x + \mathbb{Q}$. But V has only one point from each coset of $\mathbb{Q}$, so we conclude that $v = v'$, and hence $q = q'$. This proves that the sets $\{q + V \mid q \in \mathbb{Q}\}$ are pairwise disjoint.

Next, observe that for any $x \in \mathbb{R}$ there exists a point $v \in V$ so that $v \in x + \mathbb{Q}$. Then $v = x + q$ for some $q \in \mathbb{Q}$, so $x = (-q) + v$, and hence $x \in (-q) + V$. It follows that $\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (q + V)$. ■

Lemma 4

Let $V \subseteq [0,1]$ be a Vitali set, let $C = \mathbb{Q} \cap [-1,1]$, and let

$$U = \biguplus_{q \in C} (q + V).$$

Then

$$[0,1] \subseteq U \subseteq [-1,2].$$

PROOF First, since $V \subseteq [0,1]$, we know that $q + V \subseteq [-1,2]$ for all $q \in [-1,1]$ and hence $U \subseteq [-1,2]$. To prove that $[0,1] \subseteq U$, let $x \in [0,1]$. Since V is a Vitali set, there exists a $v \in V$ so that $v \in x + \mathbb{Q}$. Then $v = x + q$ for some $q \in \mathbb{Q}$. But v and x both lie in $[0,1]$, so it follows that $q = v - x$ lies in the interval $[-1,1]$. Thus $q \in C$ and $x \in q + V$, which proves that $x \in U$. ■

PROOF OF THEOREM 2 Let V be a Vitali set, and suppose to the contrary that V is measurable. Let $C = \mathbb{Q} \cap [-1,1]$, and let $U = \biguplus_{q \in C}(q + V)$. Then U is a countable union of measurable sets, and is hence measurable. By the Lemma 1, we know that

$$[0,1] \subseteq U \subseteq [-1,2].$$

and therefore $1 \leq m(U) \leq 3$. But

$$m(U) = m\left(\biguplus_{q \in C} (q + V)\right) = \sum_{q \in C} m(q + V) = \sum_{q \in C} m(V).$$

If $m(V) = 0$, then it follows that $m(U) = 0$, and if $m(V) > 0$, then it follows that $m(U) = \infty$, both of which contradict the statement that $1 \leq m(U) \leq 3$. ■

It follows from this theorem that Lebesgue outer measure m^* is not even finitely additive. In particular, recall from the homework that any set $E \subseteq [0,1]$ satisfying

$$m^*(E) + m^*\big([0,1] - E\big) = 1$$

is Lebesgue measurable. It follows that

$$m^*(V) + m^*\big([0,1] - V\big) \neq 1$$

for any Vitali set V.

As we discussed previously, a set V of finite outer measure is measurable if and only if $m_*(V) = m^*(V)$, where m_* is the Lebesgue inner measure. Since a Vitali set V is not measurable, these two quantities must in fact be different. The following proposition clarifies the situation.

Proposition 5

If V is a Vitali set then $m_*(V) = 0$ and $m^*(V) > 0$.

PROOF Let V be a Vitali set, le $C = \mathbb{Q} \cap [-1,1]$, and let $U = \biguplus_{q\in C}(q+V)$. By Lemma 1,we know that $[0,1] \subseteq U \subseteq [-1,2]$, so $1 \leq m_*(U) \leq m^*(U) \leq 3$. But

$$m^*(U) \;\leq\; \sum_{q\in C} m^*(q+V) \;=\; \sum_{q\in C} m^*(V)$$

and it follows that $m^*(V) > 0$.

As for the inner measure, recall that m_* is countably superadditive, i.e.

$$m_*\left(\biguplus_{n\in\mathbb{N}} S_n\right) \;\geq\; \sum_{n\in\mathbb{N}} m_*(S_n)$$

for any sequence $\{S_n\}$ of disjoint subsets of $\mathbb{R}$. It follows that

$$m_*(U) \;\geq\; \sum_{q\in C} m_*(q+V) \;=\; \sum_{q\in C} m_*(V),$$

and hence $m_*(V) = 0$. ■

Of course, this proposition doesn't tell us what the outer measure measure $m^*(V)$ of a Vitali set V actually is. It turns out that it depends on the Vitali set: though we will not prove it here, it is known that for any $r \in (0,1]$ there exists a Vitali set $V \subseteq [0,1]$ such that $m^*(V) = r$.

As mentioned previously, our construction of a non-measurable set depends critically on the axiom of choice. Indeed, Robert Solovay proved in 1970 that it is impossible to construct a non-measurable set without the axiom of choice. That is, Solovay proved that the statement "every subset of $\mathbb{R}$ is Lebesgue measurable" is consistent with the ZF (Zermelo-Fraenkel) axioms of set theory, i.e. all the axioms of ZFC minus the axiom of choice. Thus the axiom of choice is required for the construction of any non-measurable set.

$\mathbb{R}$ as a Vector Space over $\mathbb{Q}$

The partition of $\mathbb{R}$ into cosets of $\mathbb{Q}$ that we have been exploiting is essentially a manifestation of the fact that the rational numbers $\mathbb{Q}$ are an additive subgroup of the real numbers $\mathbb{R}$. In this section, we show how to increase the power of this technique by viewing $\mathbb{R}$ as a vector space over $\mathbb{Q}$. First, recall the following definition.

Definition: Vector Space

Let $\mathbb{F}$ be a field. A **vector space** over $\mathbb{F}$ is an abelian group $(V, +)$ together with an operation

$$\mathbb{F} \times V \to V, \qquad \text{denoted } (\lambda, v) \mapsto \lambda v$$

called **scalar multiplication**, satisfying the following axioms:

1. $\lambda(\mu v) = (\lambda\mu)v$ for all $\lambda, \mu \in \mathbb{F}$ and $v \in V$.
2. $\lambda(v + w) = \lambda v + \lambda w$ for all $\lambda \in \mathbb{F}$ and $v, w \in V$.
3. $(\lambda + \mu)v = \lambda v + \mu v$ for all $\lambda, \mu \in \mathbb{F}$ and $v \in V$.
4. $1v = v$ for all $v \in V$, where 1 denotes the multiplicative identity of $\mathbb{F}$.

Using this definition, it is not hard to prove that **the real numbers $\mathbb{R}$ form a vector space over $\mathbb{Q}$**, where the scalar multiplication function

$$\mathbb{Q} \times \mathbb{R} \to \mathbb{R}$$

is simply the usual multiplication of a rational number and a real number. All four of the axioms for a vector space are immediate.

As we shall see, this structure has many surprising consequences for $\mathbb{R}$. Before we prove anything, though, we must consider what various standard notions from linear algebra mean in this context.

Definition: Linear Independence

Let V be a vector space over a field $\mathbb{F}$, and let $I \subseteq V$. We say that I is **linearly independent** (over $\mathbb{F}$) if

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0 \qquad \Rightarrow \qquad \lambda_1 = \cdots = \lambda_n = 0$$

for every finite subset $\{v_1, \ldots, v_n\}$ of I and all $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.

It is quite possible for a subset of $\mathbb{R}$ to be linearly independent over $\mathbb{Q}$. For example, if α is an irrational number, then the set $\{1, \alpha\}$ is linearly independent over $\mathbb{Q}$. For if

$$q_1(1) + q_2\alpha \,=\, 0$$

for some rational numbers q_1, q_2, then clearly q_2 must be zero, since otherwise we would have $\alpha = -q_1/q_2$, and it follows that q_1 is zero as well.

Similar arguments can be used to prove, say, that the set $\{1, \sqrt{2}, \sqrt{3}\}$ is linearly independent over $\mathbb{Q}$. That is, if

$$q_1 + q_2\sqrt{2} + q_3\sqrt{3} \,=\, 0$$

for some rational numbers q_1, q_2, q_3, then it follows that $q_1 = q_2 = q_3 = 0$. Interestingly, it is an open question whether the set $\{1, \pi, e\}$ is linearly independent over $\mathbb{Q}$. Indeed, it is not even known whether $\pi + e$ is rational.

Definition: Basis
Let V be a vector space over a field $\mathbb{F}$. A subset $B \subseteq V$ is said to be a **basis** for V (over $\mathbb{F}$) if B is linearly independent and every element of V can be written as a finite linear combination of elements of B.

If B is a basis for V, then every nonzero $v \in V$ can be expressed *uniquely* as a finite linear combination of some elements of B with nonzero coefficients.

Now the question arises whether $\mathbb{R}$ might have a basis over $\mathbb{Q}$. Such a basis would be a set B of real numbers such that every nonzero real number could be written *uniquely* as

$$q_1 b_1 + \cdots + q_n b_n$$

for some finite subset $\{b_1, \ldots, b_n\} \subseteq B$ and some nonzero $q_1, \ldots, q_n \in \mathbb{Q}$.

It turns out that $\mathbb{R}$ does have a basis over $\mathbb{Q}$, though such a basis requires the axiom of choice to construct.

Theorem 6 Existence of Bases

Let V be a vector space over a field $\mathbb{F}$. Then there exists a basis for V. Indeed, for any linearly independent set $I \subseteq V$, there exists a basis B for V that contains I.

PROOF This theorem requires the axiom of choice, and indeed is equivalent to the axiom of choice over ZF. See Chapter III, Section 5 of Lang's *Algebra* for a complete proof. ■

The existence of a basis for $\mathbb{R}$ over $\mathbb{Q}$ has many unexpected consequences. We give two examples.

Proposition 7

There exists an uncountable set of irrational numbers that is closed under addition.

PROOF Since $\{1\}$ is linearly independent over $\mathbb{Q}$, it follows from Theorem 6 that there exists a basis B for $\mathbb{R}$ over $\mathbb{Q}$ that contains 1. It is easy to prove that B must be uncountable, since the set of finite linear combinations of the members of any countable set is countable.

Let S be the set of all real numbers of the form

$$q_0 + q_1 b_1 + \cdots + q_n b_n$$

where $\{b_1, \ldots, b_n\}$ is a finite subset of $B - \{1\}$ (with $n \geq 1$) and $q_0, q_1, \ldots, q_n$ are positive rational numbers. Then each element of S is irrational, and it is easy to see that S is both uncountable and closed under addition. ■

Incidentally, the set S that we constructed in the previous proposition has an unexpected extra property: every real number can be expressed as a difference $s_1 - s_2$ for some $s_1, s_2 \in S$.

Proposition 8

The additive groups of $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic. That is, there exists a bijection $f\colon \mathbb{R} \to \mathbb{R}^2$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

PROOF Let B be a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$, and let

$$B' = \{(b,0) \mid b \in B\} \cup \{(0,b) \mid b \in B\}.$$

Then clearly B' is a basis for $\mathbb{R}^2$ over $\mathbb{Q}$. However,

$$|B'| = |\{0,1\} \times B| = |B|$$

where the latter equality follows from the well-known fact that $|\{0,1\} \times S| = |S|$ for any infinite set S. Thus there exists a bijection $g\colon B \to B'$. Let $f\colon \mathbb{R} \to \mathbb{R}^2$ be the function defined by $f(0) = (0,0)$ and

$$f(q_1 b_1 + \cdots + q_n b_n) = q_1\, g(b_1) + \cdots + q_n\, g(b_n)$$

for any finite subset $\{b_1, \ldots, b_n\}$ of B and any $q_1, \ldots, q_n \in \mathbb{Q} - \{0\}$. Then it is easy to check that f is a bijection, and that it satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. ■

Similar arguments can be used to show that the additive groups of $\mathbb{R}^m$ and $\mathbb{R}^n$ are isomorphic for all $m, n \in \mathbb{N}$.

Complementary Subgroups to $\mathbb{Q}$

We are now ready to give another example of a non-measurable set. First recall that a nonempty set $S \subseteq \mathbb{R}$ is called a **subgroup** of $\mathbb{R}$ if it is closed under addition and negation.

Definition: Complementary Subgroups to $\mathbb{Q}$
A subgroup S of $\mathbb{R}$ is said to be **complementary to $\mathbb{Q}$** if S contains exactly one element from each coset of $\mathbb{Q}$ in $\mathbb{R}$.

That is, S is complementary to $\mathbb{Q}$ if and only if every $x \in \mathbb{R}$ can be written uniquely as $s + q$ for some $s \in S$ and $q \in \mathbb{Q}$. From a group-theoretic point of view, this is equivalent to saying that $\mathbb{R}$ is the internal direct sum of S and $\mathbb{Q}$.

Proposition 9

There exists a subgroup S of $\mathbb{R}$ that is complementary to $\mathbb{Q}$.

PROOF Let B be a basis for $\mathbb{R}$ over $\mathbb{Q}$ that contains 1, and let S be the set of all real numbers that can be written as a finite linear combination of elements of $B - \{1\}$. Then S is clearly a subgroup of $\mathbb{R}$, and it is easy to see that S must be complementary to $\mathbb{Q}$. ■

Proposition 10 A Non-Measurable Subgroup

Let S be a subgroup of $\mathbb{R}$. If S is complementary to $\mathbb{Q}$, then S is not Lebesgue measurable.

PROOF Suppose to the contrary that S is Lebesgue measurable, and let

$$U = \biguplus_{n \in \mathbb{Z}} (n + S).$$

The U should be Lebesgue measurable as well. We claim that $U \cap [0, 1)$ is a Vitali set.

Let $x + \mathbb{Q}$ be a coset of $\mathbb{Q}$ in $\mathbb{R}$. Since S is complementary to $\mathbb{Q}$, it intersects $x + \mathbb{Q}$ at a unique point s. Then

$$(x + \mathbb{Q}) \cap (n + S) = \{n + s\}$$

for each $n \in \mathbb{Z}$, so

$$(x + \mathbb{Q}) \cap U \;=\; s + \mathbb{Z}.$$

But $s + \mathbb{Z}$ intersects the interval $[0,1)$ at a single point, and therefore $x + \mathbb{Q}$ intersects $U \cap [0,1)$ at a single point, which proves that $U \cap [0,1)$ is a Vitali set. Then $U \cap [0,1)$ is not Lebesgue measurable, a contradiction. ■

Exercises

1. Prove that for any set $S \subseteq \mathbb{R}$ there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ with the property that $f(I) = S$ for every open interval I.

2. Prove that for every $\epsilon > 0$ there exists a Vitali set V such that $m^*(V) < \epsilon$.

3. Prove that the interval $[0,1]$ is a countable union of Vitali sets.

4. Prove that there exists a subset $S \subseteq \mathbb{R}$ with the following properties:

   (i) S is dense in $\mathbb{R}$, and

   (ii) S contains exactly one point from each coset of $\mathbb{Q}$ in $\mathbb{R}$.

5. Prove that the Cantor set $C \subseteq [0,1]$ does not contain a Vitali set.

6. Prove that the set $\{1, \sqrt{2}, \sqrt{3}\}$ is linearly independent over $\mathbb{Q}$.