Appendix: Norms and Inner Products

In these notes we discuss two different structures that can be put on vector spaces: norms and inner products. For the purposes of these notes, all vector spaces are assumed to be over the real numbers.

Normed Vector Spaces

Definition: Norm

Let V be a vector space. A **norm** on V is a function $\|-\|\colon V \to \mathbb{R}$ satisfying the following conditions:

1. $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in V$, and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $\|\lambda\mathbf{v}\| = |\lambda|\,\|\mathbf{v}\|$ for all $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$.
3. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ for all $\mathbf{v}, \mathbf{w} \in V$.

If $\|-\|$ is a norm on V, then the pair $(V, \|-\|)$ is called a **normed vector space**.

The first condition is sometimes called **positive definiteness**. The third condition is the **triangle inequality**.

We begin by proving some elementary statements about any normed vector space.

Proposition 1 Reverse Triangle Inequality

Let V be a normed vector space. Then

$$\|\mathbf{v} - \mathbf{w}\| \;\geq\; \|\mathbf{v}\| - \|\mathbf{w}\|$$

for all $\mathbf{v}, \mathbf{w} \in V$.

PROOF By the triangle inequality,

$$\|\mathbf{v}\| \;=\; \|(\mathbf{v} - \mathbf{w}) + \mathbf{w}\| \;\leq\; \|\mathbf{v} - \mathbf{w}\| + \|\mathbf{w}\|,$$

and the desired conclusion follows. ■

Definition: Unit Vector
Let V be a normed vector space. A vector $\mathbf{v} \in V$ is called a **unit vector** if $\|\mathbf{v}\| = 1$.

Proposition 2 Normalization

Let V be a normed vector space, and let $\mathbf{v}$ be a nonzero vector in V. Then the vector

$$\hat{\mathbf{v}} \;=\; \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

is a unit vector.

PROOF We have

$$\|\hat{\mathbf{v}}\| \;=\; \left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\| \;=\; \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| \;=\; 1.$$ ■

The vector $\hat{\mathbf{v}}$ defined above is sometimes called the **normalization** of $\mathbf{v}$. Note that

$$\mathbf{v} \;=\; \|\mathbf{v}\|\,\hat{\mathbf{v}}$$

and hence every vector in V is a scalar multiple of a unit vector.

For the following proposition, recall that a **metric** on a set X is a function

$$d\colon X \times X \to \mathbb{R}$$

satisfying the following conditions:

1. $d(x,y) \geq 0$ for all $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$ for all $x, y \in X$.
3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Note that these conditions are very similar to those in the definition of a norm, and indeed a norm can be viewed as the most natural form of metric on a vector space.

Proposition 3 Metric from a Norm

Let V be a vector space, let $\|-\|$ be a norm on V, and let $d\colon V \times V \to \mathbb{R}$ be the function

$$d(\mathbf{v}, \mathbf{w}) \;=\; \|\mathbf{v} - \mathbf{w}\|$$

Then d is a metric on V.

PROOF For condition (1), we clearly have $d(\mathbf{v},\mathbf{w}) \geq 0$ for all $\mathbf{v},\mathbf{w} \in V$, and $d(\mathbf{v},\mathbf{v}) = \|\mathbf{v}-\mathbf{v}\| = \|\mathbf{0}\| = 0$. Moreover, if $d(\mathbf{v},\mathbf{w}) = 0$, then $\|\mathbf{v}-\mathbf{w}\| = 0$. By the first condition for a norm, it follows that $\mathbf{v}-\mathbf{w} = \mathbf{0}$, and hence $\mathbf{v} = \mathbf{w}$. For condition (2), if $\mathbf{v},\mathbf{w} \in V$, then

$$d(\mathbf{w},\mathbf{v}) \;=\; \|\mathbf{w}-\mathbf{v}\| \;=\; \|(-1)(\mathbf{v}-\mathbf{w})\| \;=\; |-1|\,\|\mathbf{v}-\mathbf{w}\| \;=\; \|\mathbf{v}-\mathbf{w}\| \;=\; d(\mathbf{v},\mathbf{w}).$$

Finally, for condition (3), we have

$$\begin{aligned} d(\mathbf{u},\mathbf{w}) \;=\; \|\mathbf{u}-\mathbf{w}\| \;=\; \|(\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w})\| \\ \leq\; \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\| \;=\; d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}). \end{aligned}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in V$. ■

If X is a set and d is a metric on X, then the pair (X,d) is called a **metric space**. According to the above definition, every normed vector space is automatically a metric space, which lets us define notions such as continuity and convergence.

Finally, there is a natural notion of equivalence for normed metric space. For the following definition, recall that an **isomorphism** between two vector spaces V and W is any bijective linear transformation $V \to W$. Two vector spaces V and W are **isomorphic** if there exists an isomorphism between them, which occurs if and only if V and W have the same dimension.

Definition: Isometric Isomorphism
Let V and W be normed vector spaces. An isomorphism $T\colon V \to W$ is said to be **isometric** if

$$\|T(\mathbf{v})\| \;=\; \|\mathbf{v}\|$$

for all $\mathbf{v} \in V$. We say that V and W are **isometrically isomorphic** if there exists an isometric isomorphism from V to W.

In general, a bijection $f\colon X \to Y$ between two metric spaces X and Y is said to be **isometric** if

$$d\big(f(x_1), f(x_2)\big) \;=\; d(x_1, x_2)$$

for all $x_1, x_2 \in X$. For an isomorphism between normed vector spaces, this is equivalent to the condition given above.

Inner Product Spaces

Definition: Inner Product
Let V be a vector space. An **inner product** on V is a function $\langle -,- \rangle\colon V \times V \to \mathbb{R}$ satisfying the following conditions:

1. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in V$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ for all $\mathbf{v}, \mathbf{w} \in V$.
3. $\langle \mathbf{v}, \lambda\mathbf{w} \rangle = \lambda\langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{R}$.
4. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

If $\langle -,- \rangle$ is an inner product on V, then the pair $(V, \langle -,- \rangle)$ is called an **inner product space**.

Note that combining conditions (2) and (3) gives the equation

$$\langle \lambda\mathbf{v}, \mathbf{w} \rangle = \lambda\langle \mathbf{v}, \mathbf{w} \rangle$$

for all $\lambda \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$. Similarly, combining conditions (2) and (4) gives the equation

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle \;=\; \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

Note also that

$$\langle \mathbf{v}, \mathbf{0} \rangle \;=\; \langle \mathbf{v}, 0\mathbf{v} \rangle \;=\; 0\langle \mathbf{v}, \mathbf{v} \rangle \;=\; 0$$

for any vector $\mathbf{v} \in V$.

Definition: Associated Norm
If V is an inner product space, the **associated norm** on V is the function $\|{-}\|\colon V \to \mathbb{R}$ defined by

$$\|\mathbf{v}\| \;=\; \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

It is not immediately clear that the associated norm is actually a norm. In particular, it is by no means obvious that the triangle inequality

$$\sqrt{\langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w} \rangle} \;\leq\; \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} + \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}$$

holds for any inner product $\langle -, - \rangle$. We will prove this below, but in the meantime we will use the notation $\|\mathbf{v}\|$ to mean $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$, without making the assumption that $\|{-}\|$ satisfies the triangle inequality.

Proposition 4 Square Formulas

Let V be an inner product space. Then for any $\mathbf{v}, \mathbf{w} \in V$,

$$\|\mathbf{v}+\mathbf{w}\|^2 \;=\; \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2.$$

and

$$\|\mathbf{v}-\mathbf{w}\|^2 \;=\; \|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2.$$

PROOF We have

$$\begin{aligned} \|\mathbf{v}+\mathbf{w}\|^2 \;&=\; \langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w} \rangle \;=\; \langle \mathbf{v}+\mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{v}+\mathbf{w}, \mathbf{w} \rangle \\ &=\; \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \;=\; \langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \\ &=\; \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2. \end{aligned}$$

The second formula follows by substituting $-\mathbf{w}$ for $\mathbf{w}$. ■

Corollary 5 Pythagorean Theorem

Let V be an inner product space. If $\mathbf{v}, \mathbf{w} \in V$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, then

$$\|\mathbf{v} + \mathbf{w}\|^2 \;=\; \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

Definition: Parallel and Orthogonal Vectors
Let V be an inner product space, and let $\mathbf{w} \in V$ be a nonzero vector.

1. We say that a vector $\mathbf{v} \in V$ is **parallel** to $\mathbf{w}$ if $\mathbf{v} = \lambda \mathbf{w}$ for some $\lambda \in \mathbb{R}$.

2. We say that a vector $\mathbf{v} \in V$ is **orthogonal** to $\mathbf{w}$ if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Proposition 6 Orthogonal Decomposition

Let V be an inner product space, and let $\mathbf{w} \in V$ be a nonzero vector. Then any vector $\mathbf{v} \in V$ can be written uniquely as a sum

$$\mathbf{v} \;=\; \mathbf{p} + \mathbf{n}$$

where $\mathbf{p}$ is parallel to $\mathbf{w}$ and $\mathbf{n}$ is orthogonal to $\mathbf{w}$.

PROOF Let $\mathbf{v} \in V$, and let

$$\mathbf{p} \;=\; \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w} \qquad \text{and} \qquad \mathbf{n} = \mathbf{v} - \mathbf{p}$$

so $\mathbf{v} = \mathbf{p} + \mathbf{n}$. Clearly $\mathbf{p}$ is parallel to $\mathbf{w}$, and

$$\langle \mathbf{p}, \mathbf{w} \rangle \;=\; \left\langle \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}, \mathbf{w} \right\rangle \;=\; \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\langle \mathbf{w}, \mathbf{w} \rangle \;=\; \langle \mathbf{v}, \mathbf{w} \rangle$$

so

$$\langle \mathbf{n}, \mathbf{w} \rangle \;=\; \langle \mathbf{v} - \mathbf{p}, \mathbf{w} \rangle \;=\; \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{p}, \mathbf{w} \rangle \;=\; 0.$$

and hence $\mathbf{n}$ is orthogonal to $\mathbf{w}$.

To prove this decomposition is unique, suppose $\mathbf{p} = \lambda \mathbf{w}$ is any vector parallel to $\mathbf{w}$ and $\mathbf{n}$ is any vector orthogonal to $\mathbf{w}$ such that $\mathbf{v} = \mathbf{p} + \mathbf{n}$. Then

$$0 \;=\; \langle \mathbf{n}, \mathbf{w} \rangle \;=\; \langle \mathbf{v} - \mathbf{p}, \mathbf{w} \rangle \;=\; \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{p}, \mathbf{w} \rangle \;=\; \langle \mathbf{v}, \mathbf{w} \rangle - \lambda \langle \mathbf{w}, \mathbf{w} \rangle.$$

It follows that $\lambda = \langle \mathbf{v}, \mathbf{w} \rangle / \langle \mathbf{w}, \mathbf{w} \rangle$, which means that $\mathbf{p}$ is the same as the vector given above. It follows that $\mathbf{n} = \mathbf{v} - \mathbf{p}$ is the same as well. ■

The vector

$$\mathbf{p} \;=\; \frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}\mathbf{w}$$

from the previous proposition is usually called the **projection** of $\mathbf{v}$ onto $\mathbf{w}$.

Theorem 7 Cauchy-Schwarz Inequality

If V is an inner product space, then

$$|\langle \mathbf{v}, \mathbf{w}\rangle| \;\leq\; \|\mathbf{v}\|\,\|\mathbf{w}\|$$

for all $\mathbf{v}, \mathbf{w} \in V$.

PROOF If $\mathbf{w} = \mathbf{0}$ then the inequality clearly holds, so suppose that $\mathbf{w} \neq \mathbf{0}$. Then

$$\mathbf{v} \;=\; \mathbf{p} + \mathbf{n},$$

where $\mathbf{p}$ is the projection of $\mathbf{v}$ onto $\mathbf{w}$, and $\mathbf{n}$ is orthogonal to $\mathbf{w}$. By the Pythagorean theorem,

$$\|\mathbf{v}\|^2 \;=\; \|\mathbf{p}\|^2 + \|\mathbf{n}\|^2$$

and hence $\|\mathbf{v}\| \geq \|\mathbf{p}\|$. But since $\mathbf{p}$ is parallel to $\mathbf{w}$, we know that $\mathbf{p} = \lambda\mathbf{w}$ for some $\lambda \in \mathbb{R}$, and thus

$$\langle \mathbf{p}, \mathbf{w}\rangle \;=\; \lambda\langle \mathbf{w}, \mathbf{w}\rangle \;=\; \lambda\|\mathbf{w}\|^2 \;=\; \|\mathbf{p}\|\,\|\mathbf{w}\|.$$

Then

$$\langle \mathbf{v}, \mathbf{w}\rangle \;=\; \langle \mathbf{p}, \mathbf{w}\rangle \;=\; \|\mathbf{p}\|\,\|\mathbf{w}\| \;\leq\; \|\mathbf{v}\|\,\|\mathbf{w}\|.$$

■

Theorem 8 A Norm from an Inner Product

Let V be a vector space, and let $\langle -, - \rangle$ be an inner product on V. Then the function $\|-\|\colon V \to \mathbb{R}$ defined by

$$\|\mathbf{v}\| \;=\; \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

is a norm on V.

PROOF Clearly $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in V$, with $\|\mathbf{0}\| = \sqrt{\langle \mathbf{0}, \mathbf{0}\rangle} = \sqrt{0} = 0$. Moreover, if $\mathbf{v} \in V$ and $\|\mathbf{v}\| = 0$, then $\langle \mathbf{v}, \mathbf{v}\rangle = 0$, and it follows that $\mathbf{v} = \mathbf{0}$. Next, if $\mathbf{v} \in V$ and

$\lambda \in \mathbb{R}$, then

$$\|\lambda \mathbf{v}\| \;=\; \sqrt{\langle \lambda \mathbf{v}, \lambda \mathbf{v} \rangle} \;=\; \sqrt{\lambda^2 \langle \mathbf{v}, \mathbf{v} \rangle} \;=\; \sqrt{\lambda^2}\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \;=\; |\lambda|\,\|\mathbf{v}\|.$$

Finally, if $\mathbf{v}, \mathbf{w} \in V$, then by the Cauchy-Schwarz inequality

$$\langle \mathbf{v}, \mathbf{w} \rangle \;\leq\; \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}\sqrt{\langle \mathbf{w}, \mathbf{w} \rangle} \;=\; \|\mathbf{v}\|\|\mathbf{w}\|$$

so by the square formula

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 \;&=\; \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \\ &\leq\; \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 \;=\; \big(\|\mathbf{v}\| + \|\mathbf{w}\|\big)^2, \end{aligned}$$

and hence $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$. ■

Recovering the Inner Product

So far we have shown that an inner product on a vector space always leads to a norm. The following proposition shows that we can get the inner product back if we know the norm.

Proposition 9 Polarization Identity

Let V be a vector space, let $\langle -, - \rangle$ be an inner product on V, and let $\|-\|$ be the corresponding norm. Then for any $\mathbf{v}, \mathbf{w} \in V$,

$$\langle \mathbf{v}, \mathbf{w} \rangle \;=\; \frac{\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2}{4}$$

PROOF This follows immediately from the square formulas in Proposition 4. ■

As a consequence of the polarization identity, we obtain a characterization of isometric isomorphisms between inner product spaces.

Proposition 10 Isometric Isomorphisms and Inner Products

Let V and W be inner product spaces. Then an isomorphism $T\colon V \to W$ is isometric if and only if

$$\big\langle T(\mathbf{v}_1), T(\mathbf{v}_2)\big\rangle \;=\; \langle \mathbf{v}_1, \mathbf{v}_2\rangle$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$.

PROOF Suppose first that the given identity holds. Then

$$\|T(\mathbf{v})\| \;=\; \sqrt{\big\langle T(\mathbf{v}), T(\mathbf{v})\big\rangle} \;=\; \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} \;=\; \|\mathbf{v}\|$$

for all $\mathbf{v} \in V$, and hence T is isometric. For the converse, suppose that T is isometric, and let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then by the polarization identity,

$$\begin{aligned}\big\langle T(\mathbf{v}_1), T(\mathbf{v}_2)\big\rangle \;&=\; \frac{\|T(\mathbf{v}_1)+T(\mathbf{v}_2)\|^2 + \|T(\mathbf{v}_1)-T(\mathbf{v}_2)\|^2}{4} \\ &=\; \frac{\|T(\mathbf{v}_1+\mathbf{v}_2)\|^2 + \|T(\mathbf{v}_1-\mathbf{v}_2)\|^2}{4} \\ &=\; \frac{\|\mathbf{v}_1+\mathbf{v}_2\|^2 + \|\mathbf{v}_1-\mathbf{v}_2\|^2}{4} \;=\; \langle \mathbf{v}_1, \mathbf{v}_2\rangle \qquad \blacksquare\end{aligned}$$

Since the polarization identity allows us to recover the inner product from the norm, a natural question is whether *any* norm can be used to define an inner product via the polarization identity. The answer to this question is no, as suggested by the following proposition.

Proposition 11 Parallelogram Law

Let V be a vector space, let $\langle -,-\rangle$ be an inner product on V, and let $\|-\|$ be the corresponding norm. Then

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \;=\; \frac{\|\mathbf{v}+\mathbf{w}\|^2 + \|\mathbf{v}-\mathbf{w}\|^2}{2}$$

for all $\mathbf{v}, \mathbf{w} \in V$.

PROOF Again, this follows immediately from the two square formulas given in Proposition 4. ■

There is no reason that an arbitrary norm would obey the parallelogram law, and hence most norms do not correspond to an inner product. For example, it is easy to check that the p-norm on $\mathbb{R}^n$ obeys the parallelogram law if and only if $p = 2$, and thus the Euclidean norm is the only p-norm that can be obtained from an inner product.

Incidentally, it is possible to prove that any norm that obeys the parallelogram law *can* be derived from an inner product. See `http://math.stackexchange.com/questions/21792`.

Orthonormal Bases

Definition: Orthonormal Vectors, Orthonormal Basis
Let V be an inner product space, and let U be a set of vectors in V. We say that the vectors in U are **orthonormal** if every vector in U is a unit vector and every pair of distinct vectors in U are orthogonal. If U is also a basis for V, then U is called an **orthonormal basis** for V.

It is easy to prove that any orthonormal set U of vectors must be linearly independent (see Exercise 14).

Proposition 12 Existence of Orthonormal Bases

Every finite-dimensional vector space has an orthonormal basis.

PROOF Let V be a finite-dimensional inner product space of dimension n. We proceed by induction on n. If $n = 0$, then the empty set is a basis for V, and clearly this is orthonormal.

For $n \geq 1$, let $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be any basis for V. By our induction hypothesis, the $(n-1)$-dimensional subspace $S = \mathrm{Span}\{b_1, \ldots, b_{n-1}\}$ has an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$. Let $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$ be the projections of $\mathbf{b}_n$ onto $\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$, respectively, and let $\mathbf{p}_n = \mathbf{b}_n - (\mathbf{p}_1 + \cdots + \mathbf{p}_{n-1})$. Since $\mathbf{p}_1, \ldots \mathbf{p}_{n-1} \in S$ and $\mathbf{b}_n \notin S$, we know that $\mathbf{p}_n \neq 0$. Let $\mathbf{u}_n = \mathbf{p}_n / \|\mathbf{p}_n\|$ be the normalization of $\mathbf{p}_n$. Then $\mathbf{u}_n$ is a unit vector and

$$\begin{aligned}\langle \mathbf{u}_i, \mathbf{u}_n \rangle &= \big\langle \mathbf{u}_i, \mathbf{b}_n - (\mathbf{p}_1 + \cdots + \mathbf{p}_{n-1}) \big\rangle \\ &= \langle \mathbf{u}_i, \mathbf{b}_n \rangle - \big(\langle \mathbf{u}_i, \mathbf{p}_1 \rangle + \cdots + \langle \mathbf{u}_i, \mathbf{p}_{n-1} \rangle \big) \\ &= \langle \mathbf{u}_i, \mathbf{b}_n \rangle - \langle \mathbf{u}_i, \mathbf{p}_i \rangle \;=\; 0\end{aligned}$$

for all $i \in \{1, \ldots, n-1\}$, so the set $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}, \mathbf{u}_n\}$ is orthonormal. Since U is linearly independent and has n elements, it is a basis for V, and thus V has an orthonormal basis. ■

This theorem actually does not extend to infinite-dimensional vector spaces. That is, there exists an infinite-dimensional inner product space that does not have any orthonormal basis. For this reason we shall restrict ourselves to finite-dimensional spaces.

Proposition 13 Formulas Involving Coefficients

Let V be an finite-dimensional inner product space, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for V, and let

$$\mathbf{v} = v_1\mathbf{u}_1 + \cdots + v_n\mathbf{u}_n \qquad \textit{and} \qquad \mathbf{w} = w_1\mathbf{u}_1 + \cdots + w_n\mathbf{u}_n$$

be vectors in V. Then:

1. $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + \cdots + v_n w_n$.
2. $\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots v_n^2}$.
3. $v_i = \langle \mathbf{u}_i, \mathbf{v} \rangle$ *for each i.*

PROOF For (1), we have

$$\langle \mathbf{v}, \mathbf{w} \rangle = \left\langle \sum_{i=1}^n v_i \mathbf{u}_i,\; \sum_{j=1}^n w_j \mathbf{u}_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n v_i w_i \langle \mathbf{u}_i, \mathbf{u}_j \rangle.$$

But $\langle \mathbf{u}_i, \mathbf{u}_j \rangle$ is equal to 1 if $i = j$ and 0 otherwise, so

$$\langle \mathbf{v}, \mathbf{w} \rangle \;=\; \sum_{i=1}^{n} v_i w_i.$$

Statement (2) follows immediately from (1) and the fact that $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. Statement (3) also follows immediately from statement (1). ■

For the following theorem, recall that the **Euclidean norm** on $\mathbb{R}^n$ refers to the usual 2-norm.

Theorem 14 Structure of Finite-Dimensional Inner Product Spaces

If $n \in \mathbb{N}$, then every n-dimensional inner product space is isometrically isomorphic to $\mathbb{R}^n$ under the Euclidean norm.

PROOF Let V be an n-dimensional inner product space, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and define a function $T \colon \mathbb{R}^n \to V$ by

$$T(x_1, \ldots, x_n) \;=\; x_1 \mathbf{u}_1 + \cdots + x_n \mathbf{u}_n.$$

It is easy to check that T is linear and is a bijection. Furthermore,

$$\|T(x_1, \ldots, x_n)\| \;=\; \|x_1 \mathbf{u}_1 + \cdots + x_n \mathbf{u}_n\| \;=\; \sqrt{x_1^2 + \cdots + x_n^2} \;=\; \|(x_1, \ldots, x_n)\|$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, so T is isometric. ■

Incidentally, it is sometimes helpful to relax the conditions on orthonormal bases. If V is an inner product space, an **orthogonal basis** for V is any basis of orthogonal vectors (which may or may not be unit vectors). If $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is an orthogonal basis for V, then the normalizations

$$\left\{ \frac{\mathbf{b}_1}{\|\mathbf{b}_1\|}, \ldots, \frac{\mathbf{b}_n}{\|\mathbf{b}_n\|} \right\}$$

form an orthonormal basis for V. From this one can derive the following formulas:

1. If $\mathbf{v} = v_1 \mathbf{b}_1 + \cdots + v_n \mathbf{b}_n$ and $\mathbf{w} = w_1 \mathbf{b}_1 + \cdots + w_n \mathbf{b}_n$, then

$$\langle \mathbf{v}, \mathbf{w} \rangle \;=\; v_1 w_1 \, \|\mathbf{b}_1\|^2 \;+\; \cdots \;+\; v_n w_n \, \|\mathbf{b}_n\|^2.$$

2. If $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$, then
$$\|\mathbf{v}\| \;=\; \sqrt{v_1^2\,\|\mathbf{b}_1\|^2 \,+\, \cdots \,+\, v_n^2\,\|\mathbf{b}_n\|^2}.$$

3. If $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$, then
$$v_i = \frac{\langle \mathbf{b}_i, \mathbf{v}\rangle}{\langle \mathbf{b}_i, \mathbf{b}_i\rangle}$$
for each i. That is $\mathbf{v}$ is the sum of its projections onto $\mathbf{b}_1, \ldots, \mathbf{b}_n$:
$$\mathbf{v} \;=\; \frac{\langle \mathbf{b}_1, \mathbf{v}\rangle}{\langle \mathbf{b}_1, \mathbf{b}_1\rangle}\mathbf{b}_1 \,+\, \cdots \,+\, \frac{\langle \mathbf{b}_n, \mathbf{v}\rangle}{\langle \mathbf{b}_n, \mathbf{b}_n\rangle}\mathbf{b}_n.$$

Exercises

1. Prove that “isometrically isomorphic” is an equivalence relation on normed vector spaces.

2. Prove that any isometric isomorphism is a homeomorphism.

3. Let V and W be normed vector spaces, let $T\colon V \to W$ be a linear transformation, and suppose there exists a $\lambda > 0$ so that
$$\|T(\mathbf{v})\| \;\leq\; \lambda\,\|\mathbf{v}\|$$
for all $\mathbf{v} \in V$. Prove that T is continuous.

4. If V is a normed vector space, prove that the norm $\|-\|\colon V \to \mathbb{R}$ is a continuous function on V.

5. If V and W are normed vector spaces, prove that
$$\|(\mathbf{v}, \mathbf{w})\| \;=\; \|\mathbf{v}\| + \|\mathbf{w}\|$$
is a norm on $V \times W$.

6. If V is a normed vector space, prove that addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ are continuous functions.

7. Let V be an inner product space, and let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Prove that if
$$\langle \mathbf{v}_1, \mathbf{w}\rangle \;=\; \langle \mathbf{v}_2, \mathbf{w}\rangle$$
for all $\mathbf{w} \in V$, then $\mathbf{v}_1 = \mathbf{v}_2$.

8. If V and W are inner product spaces, prove that

$$\big\langle (\mathbf{v}_1, \mathbf{w}_1), (\mathbf{v}_2, \mathbf{w}_2) \big\rangle \;=\; \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$$

   is an inner product on $V \times W$.

9. If V is an inner product space, prove that the inner product $\langle -, - \rangle \colon V \times V \to \mathbb{R}$ is a continuous function.

10. Let V be an inner product space, let $\mathbf{v}, \mathbf{w} \in V$ be nonzero vectors, and let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto $\mathbf{w}$. Prove that $\mathbf{p}$ is the closest point in $\operatorname{Span}\{\mathbf{w}\}$ to $\mathbf{v}$.

11. Let V be an inner product space, let S be a linear subspace of V, and let

$$S^{\perp} \;=\; \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{s} \rangle = 0 \text{ for all } \mathbf{s} \in S\}.$$

    Prove that $S^{\perp}$ is a linear subspace of V.

12. Find vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$ for which

$$\|\mathbf{v}\|_1^2 \,+\, \|\mathbf{w}\|_1^2 \;\neq\; \frac{\|\mathbf{v}+\mathbf{w}\|_1^2 \,+\, \|\mathbf{v}-\mathbf{w}\|_1^2}{2},$$

    where $\|-\|_1$ denotes the 1-norm on $\mathbb{R}^2$. What does this prove about the 1-norm?

13. Let $\varparallelogram ABCD$ be a parallelogram in the Euclidean plane, where A is opposite C. Given that $AB = 5$, $BC = 5$, and $AC = 8$, use the parallelogram law to find BD.

14. Let V be an inner product space. Prove that any orthonormal set of vectors in V is linearly independent.

15. Let V and W be n-dimensional inner product spaces, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for V, and let $T \colon V \to W$ be an isometric isomorphism. Prove that $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_n)\}$ is an orthonormal basis for W.

16. Let V be the vector space of all polynomials of the form $p(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, and let $\langle -, - \rangle$ be the inner product on V defined by

$$\langle p, q \rangle \;=\; \int_0^1 p(x)\, q(x)\, dx.$$

    Find an orthonormal basis for V.