Precalculus Review

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Contents

1.1 Algebra

Calculus makes use of precalculus—hence the name of the latter—but to do precalculus, a solid knowledge of basic algebra is needed. We review here a few of the most important ideas from algebra that are needed for calculus.

Types of Numbers

Precalculus, and calculus, takes place within the context of the real numbers. Within the real numbers, there are some import special types of numbers that are frequently used in mathematics.

Types of Numbers

1. The **real numbers**, denoted $\mathbb{R}$, are all the numbers on the number line, including positive numbers, negative numbers, zero, whole numbers, fractions, and all other numbers (such as $\sqrt{2}$ and π).
2. The **rational numbers**, denoted $\mathbb{Q}$, are all numbers that are expressible as fractions, for example $\frac{2}{3}$ or -0.5.
3. The **integers**, denoted $\mathbb{Z}$, are the numbers $-4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$.
4. The **natural numbers**, also called the **positive integers**, denoted $\mathbb{N}$, are the numbers $1, 2, 3, 4, \ldots$.

Note that all natural numbers are integers, and all integers are rational numbers, and all rational numbers are real numbers, but not the other way around.

A collection of numbers that is even larger than the set of real numbers is the set of complex numbers, denoted $\mathbb{C}$. It is not assumed that the reader is familiar with the complex numbers. These numbers are not used in *Calculus I* and *Calculus II*; they do arise in *Introduction to Linear Algebra and Ordinary Differential Equations*, and they will be discussed there.

Infinity

We will, at times, be using the symbols ∞ and $-\infty$ to denote "infinity" and "negative infinity," respectively. These words are written in quotes to emphasize the following.

☞ **Error Warning** The symbols ∞ and $-\infty$ are not numbers. These symbols represent what happens as we take numbers that get larger and larger without bound (going to ∞) and get smaller and smaller (meaning negative numbers having larger and larger magnitude).

For example, the numbers $2, 4, 8, 16, 32, \ldots$ are "going to ∞," and the numbers $-1, -3, -5, -7, -9, \ldots$ are "going to $-\infty$."

☞ **Error Warning** Do not try to use the symbols ∞ and $-\infty$ in algebraic expressions (for example "$\infty + 5$").

Intervals

Intervals are a very useful type of collections of real numbers. An interval is the set of all numbers between two fixed numbers, where the endpoints might or might not be included in the interval. The different types of interval are as follows.

Intervals

Let a and b be real numbers. Suppose that $a \leq b$.

Notation	Type of Interval	Definition
(a, b)	**open bounded interval**	$a < x < b$
$[a, b]$	**closed bounded interval**	$a \leq x \leq b$
$[a, b)$	**half-open interval**	$a \leq x < b$
$(a, b]$	**half-open interval**	$a < x \leq b$
(a, ∞)	**open unbounded interval**	$a < x$
$(-\infty, b)$	**open unbounded interval**	$x < b$
$(-\infty, \infty)$	**open unbounded interval**	all real numbers
$[a, \infty)$	**closed unbounded interval**	$a \leq x$
$(-\infty, b]$	**closed unbounded interval**	$x \leq b$

For example, the interval $[2, 5]$ is the set of all real numbers x such that $2 \leq x \leq 5$. The interval $(3, \infty)$ is the set of all real numbers x such that $3 < x$.

☞ **Error Warning** The notation (a, b), for example $(1, 6)$, is used to mean different things in mathematics. In the present context the notation $(1, 6)$ means the interval from 1 to 6, not including the endpoints. On the other hand, when discussing points in the plane (usually denoted $\mathbb{R}^2$), the notation $(1, 6)$ means the point in $\mathbb{R}^2$ with x-coordinate 1 and y-coordinate 6. The fact that the same mathematical notation can mean very different things in different contexts can be confusing, but it is a historical accident with which we are now stuck. Fortunately, the meaning of notation such as (a, b) can usually be figured out from the context.

☞ **Error Warning** The symbols ∞ and $-\infty$ are not numbers, and cannot be included in an interval. Hence, there is no interval of the form "$[2, \infty]$."

Absolute Value

A very useful function for working with numbers is the absolute value function, which is defined as follows.

Absolute Value

Let x be a real number. The **absolute value** of x, denoted $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

The absolute value function has a number of very nice properties, including the following.

Absolute Value: Properties

Let x, y and b be real numbers.

1. $|-x| = |x|$.
2. $|x|^2 = x^2$.
3. $|x - y| = |y - x|$.
4. $|xy| = |x||y|$.
5. $|x| < b$ if and only if $-b < x < b$.

Basic Algebra Formulas

There are a few basic algebra formulas involving multiplying and factoring simple polynomials that will be useful throughout calculus.

Basic Algebra Formulas

Let a and b be real numbers.

1. $(a+b)^2 = a^2 + 2ab + b^2$.
2. $(a-b)^2 = a^2 - 2ab + b^2$.
3. $(a+b)(a-b) = a^2 - b^2$.

There are also formulas for expressions such as $(a+b)^3$ that are useful on occasion, though there is no need to remember such formulas, because they can be looked up, or worked out as needed. For example, the expression $(a+b)^3$ can be computed by rewriting it as $(a+b)^2(a+b)$, using the formula for $(a+b)^2$, and multiplying the resulting polynomials.

Solving Quadratic Equations

Solving quadratic equations is needed on occasion in calculus. Such equations can be solved in some cases by factoring (which is the quicker method when it works), and in all cases by the quadratic formula. Except for a few situation involving differential equations, we are generally interested only in solutions of equations that are real numbers, not complex numbers. We note that not every quadratic equation has solutions that are real numbers.

Solving Quadratic Equations

There are two methods to solve the equation $x^2 + bx + c = 0$.

1. If numbers r and s can be found such that $r + s = b$ and $rs = c$, then $x^2 + bx + c = (x + r)(x + s)$, and the roots of $x^2 + bx + c = 0$ are $x = -r$ and $x = -s$.

2. The roots of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided $b^2 - 4ac \geq 0$.

Fractions and Rational Expressions

A elementary topic that is needed for calculus, and that, for whatever reason, is something that not every student of calculus knows sufficiently well, is the addition, subtraction, multiplication and division of fractions. Specifically, for calculus we need to add, subtract, multiply and divide fractions that involve letters as well as numbers, and fractions that have fractions in their numerators and denominators.

One of the key idea to keep in mind in algebra is that letters in algebra simply stand for numbers that we don't know their values, and we therefore treat letters exactly the same as we would treat numbers. In particular, the familiar rules for adding, subtracting, multiplying and dividing fractions with numbers work just as well for fractions with letters, and for built-up fractions.

One other thing to keep in mind is that when dealing with built-up fractions, which have fractions in their numerators and/or denominators, it is important to distinguish the main fraction line from the subsidiary fraction lines. Visually, the best way to make this distinction is to write the main fraction line longer than the other fraction lines. Even better, the main fraction line should be written not only longer than the other fraction lines, but should be written level with the equals sign.

There are three particular types of built-up fractions that can cause confusion, and which we examine. The way to simplify these types of fractions should *not* be memorized. Rather, all such fractions should be simplified using the basic rules for adding, subtracting, multiplying and dividing fractions.

1. Simplify $\dfrac{\frac{a}{b}}{c}$.

   This fraction can be simplified by rewriting the denominator as $\frac{c}{1}$, yielding

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}.$$

2. Simplify $\dfrac{a}{\frac{b}{c}}$.

   This fraction can be simplified by rewriting the numerator as $\frac{a}{1}$, yielding

$$\frac{a}{\frac{b}{c}} = \frac{\frac{a}{1}}{\frac{b}{c}} = \frac{a}{1} \cdot \frac{c}{b} = \frac{ac}{b}.$$

3. Simplify $\dfrac{\frac{a}{b} + \frac{c}{d}}{e}$.

   This fraction can be simplified by first adding the two fractions in the numerator, and then using the method of Item 1, yielding

$$\frac{\frac{a}{b} + \frac{c}{d}}{e} = \frac{\frac{ad+bc}{bd}}{e} = \frac{\frac{ad+bc}{bd}}{\frac{e}{1}} = \frac{ad + bc}{bd} \cdot \frac{1}{e} = \frac{ad + bc}{bde}.$$

The following two examples are both used in calculus.

Example 1

Simplify $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$.

Solution We compute

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\frac{\frac{x-(x+h)}{x(x+h)}}{h}=\frac{\frac{-h}{x(x+h)}}{\frac{h}{1}}=\frac{-h}{x(x+h)}\cdot\frac{1}{h}=-\frac{1}{x(x+h)}.$$

Example 2

Simplify $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

Solution Here we use a little trick, which is

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$
$$=\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

The expression $\sqrt{x+h}+\sqrt{x}$, which is used in order to remove the square roots in the numerator, is referred to as the "conjugate" of $\sqrt{x+h}-\sqrt{x}$.

EXERCISES

1–4 ■ Multiply and then simplify each expression.

1. $(3x+5)(x^2-2x+4)$

2. $(2m+3n)(3m^2+5mn-n^2)$

3. $(2y+1)(y-5)(3y+4)$

4. $5p^2(p+3)(3p^2+4)$

5–8 ■ Multiply and then simplify each expression by using basic formulas.

5. $(5a+3)^2$

6. $(3m-4n)^2$

7. $(5y+1)(5y-1)$

8. $(2s^2-t)(2s^2+t)$

9–12 ■ Factor each expression by using basic formulas.

9. $x^2 + 8x + 16$

10. $4m^2 - 12mn + 9n^2$

11. $z^2 - 49$

12. $9y^4 - 16x^6$

13–16 ■ Factor each quadratic.

13. $x^2 - 9x + 20$

14. $x^2 - 3x - 28$

15. $2x^2 + 7x + 3$

16. $6x^2 + 11x - 10$

17–20 ■ Solve each equation.

17. $x^2 + 2x - 8 = 0$

18. $x^2 - 10x + 25 = 0$

19. $x^2 + 5x - 3 = 0$

20. $3x^2 - x - 5 = 0$

21–24 ■ Simplify each expression

21. $\dfrac{\frac{3}{x^2}}{\frac{6}{x^5}}$

22. $\dfrac{\frac{2}{a} + 1}{\frac{4}{a^2} - 1}$

23. $\dfrac{\frac{1}{n+1} - \frac{1}{n+2}}{\frac{5}{n+1} - 3}$

24. $\dfrac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

25–28 ■ Solve each equation.

25. $\dfrac{x^2 + 2x - 3}{3x^2 + 6x + 15} = 0$

26. $\dfrac{x^2 - x - 6}{x^2 - 4} = 0$

27. $\dfrac{\frac{5}{x+2} - \frac{3}{x}}{\frac{7}{x+2} - \frac{1}{x}} = 0$

28. $\dfrac{\frac{1}{x^2} - \frac{1}{5^2}}{x - 5} = 0$

1.2 Functions and Graphs

Functions are the main ingredient in calculus. The two main things we do in calculus, namely, derivatives and integrals, and things that are done to functions.

Functions are also a unifying approach in mathematics. For example, whereas logarithms and trigonometry seem to be very different, what we are interested in here is logarithmic functions and trigonometric functions, and, even though these two types of functions arise from very different considerations, as functions we treat them just as we do any other functions.

One thing to keep in mind about functions is that it is not correct to think of functions simply as formulas, for example $f(x) = x^2$. Whereas it is true that many useful functions are given by formulas, there are also useful functions that are not given by single formulas, not to mention functions not given by formulas at all. The most basic idea of a function is that it takes some sort of object as input (in calculus the input is numbers or vectors, though other types of input are used elsewhere), and for each possible input, there is one and only one output.

There are different ways of representing functions, including

1. Verbally
2. Numerically (table of values)
3. Graphically
4. By formula (some says "algebraically," but that isn't correct).

All these methods of describing a function are equivalent, and it is important to be able to go from one method to the other, for example to go from formula to graph and vice-versa.

Domain of a Function

Every function can take certain things as inputs. For example, the function $f(x)$ defined by the formula $f(x) = x^2$ can take all real numbers as inputs, whereas the function $g(x)$ defined by the formula $g(x) = \ln x$ can take only positive real numbers as inputs.

In our present context, we are considering functions with real numbers as inputs, and we then have the following concepts.

Domain and Range

Let $f(x)$ be a function with real numbers as inputs.

1. The **domain** of $f(x)$ is the set of all possible real numbers for which the function produces an output.
2. The **range** of $f(x)$ is the set of all outputs of the function, when everything in the domain of $f(x)$ is substituted into the function.

The range of a function can be useful in some contexts, though for our purpose the domain is the much more important concept.

In more advanced mathematics, the concept of the domain of a function, which takes on even more importance, is slightly more general than we are using here.

There is no definitive method for finding the domain of a function. However, there are a few things to keep in mind. For example, because it is not possible to divide by zero, we exclude anything from the domain that would lead to dividing by zero. Hence, the domain of the function defined by the formula $f(x) = \dfrac{1}{x-2}$ is the set of all real numbers other than 2.

Other standard considerations when finding the domain of a function is that we cannot take the square root of a negative number (we are considering only real numbers here); we cannot take the logarithms of a negative number or zero; and we cannot take the tangent of $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$, etc.

Substituting Numbers and Expressions into Functions

The point of a function is that we put things into it, and get something out of it for each thing we put into it. For example, let $f(x)$ be the function defined by the formula $f(x) = x^2$. Clearly, if we put 3 into the function, we get $f(3) = 9$ as the output. It can certainly happen that different inputs produce the same output. For example, we note that $f(-3) = 9$ for this function. The crucial thing to observe is that a single input produces a single output.

For example, let $g(x)$ be the function defined by the formula $g(x) = \sqrt{x}$. First, we observe that the domain of $g(x)$ is the set of all non-negative numbers. More importantly, we note that when we write $\sqrt{x}$, we mean only the positive square root of x. For example, we have $g(4) = 2$. It is certainly true that -2 is also a square root of 4, but we cannot say that $g(4)$ is ± 2, because that would give us two outputs for the single input 4. Hence, we use the standard convention that $\sqrt{x}$ always means the positive square root of x. If we want to obtain the negative square root of a number, we would need a different function, namely, the function $h(x)$ defined by the formula $h(x) = -\sqrt{x}$.

For calculus, we need to substitute not only single numbers into functions, but also more complicated expression. For example, let $f(x)$ be the function defined by the formula $f(x) = x^2$. Then we will need to compute the expression $f(x+h)$, which is given by

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2.$$

☞ **Error Warning** In the above example, it is important to recognize that $f(x+h)$ is not the same as $f(x)+h$, which would be x^2+h. That is, when we add h to x “inside” the function, that is not the same as adding h to the “outside” of the function.

Graphs of Functions

Just as a person can be encountered in different ways (for example, in person, by phone, by email, on social media), so too can a function be seen in different ways. One way of describing a function is via a formula, for example the function $f(x)$ defined by the formula $f(x) = x^2$. Another way of describing the same function is visually, via its graph.

The graph of a function of the form $y = f(x)$ is the subset of the plane consisting of all points (a,b) that satisfy the equation $f(a) = b$.

For example, for the function $f(x)$ defined by the formula $f(x) = x^2$, the point $(3,9)$ is in the graph of the function, because $f(3) = 3^2 = 9$.

To find all the points on the graph of a function, the most direct method would be to take every number in the domain of the function and put it into the function to find the output, and then plot all the points obtained in this way. Of course, doing that is not physically possible, because are infinitely many real numbers. Nonetheless, we can figure out what the graphs of many functions looks like. For example, the graph of the function $f(x)$ defined by the formula $f(x) = x^2$ is seen in Figure 1 of this section.

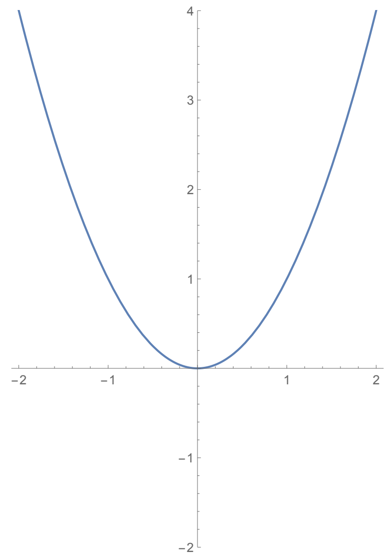


Figure 1: Graph of $f(x) = x^2$

☞ **Error Warning** There are a number of useful things to do when graphing functions, some of which you will learn in calculus, and there is one thing you should definitely not do when you graph a function: finding the value of the function at a few values of x, plotting those points in the plane, and "connecting the dots." If you could plot hundreds of very close points, you would likely have a good approximation of the graph (that's how computers plot graphs), but plotting a large number of points by hand is not practical, and plotting just a few points is not a good way to obtain an accurate picture of what the graph actually looks like, because the few points that were plotted might miss the important features of the graph. The way to plot a graph is to know what the graph ought to look like, which means to use the basic graphs that you know (lines, quadratics, trigonometric functions, exponentials, logarithms, etc.), and then modify those basic graphs, and, once you know some calculus, to use that to obtain much more information about graphs.

In the same way that anyone learning a new language needs to know some basic vocabulary and grammar by heart without having to consult a dictionary, anyone who studies calculus should know the graphs of some basic functions—without a calculator! These functions will be identified in the subsequent sections of this chapter, starting with a few given below.

Finally, we note that whereas the graph of every function of the form $y = f(x)$ is a curve in the plane, not every curve in the plane is the graph of a function. For a curve in the plane to be the graph of a function, each value of x can have either one point on the curve (if that x is in the domain of the function) or no point on the curve (if that x is not in the domain). In other words, every vertical line in the plane can intersect the curve either once or not at all; any curve that satisfies this condition with regard to vertical lines is said to pass the vertical line test, and is the graph of a function.

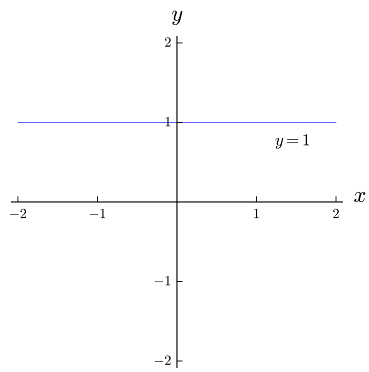
For example, a parabola which has the x-axis as its line of symmetry does not satisfy the vertical line text, and hence is not the graph of a function, whereas a parabola that has the y-axis as its line of symmetry does satisfy the vertical line text, and hence is the graph of a function

Graphs You Should Know

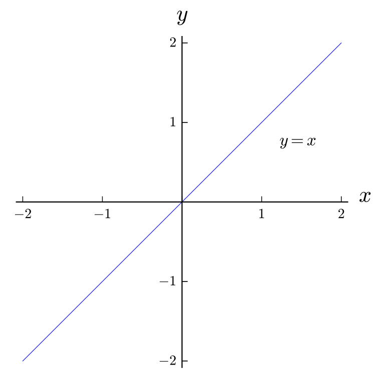
Graphs you should know: $y = c$, and $y = x$, and $y = -x$ and $y = |x|$.

Basic Graphs

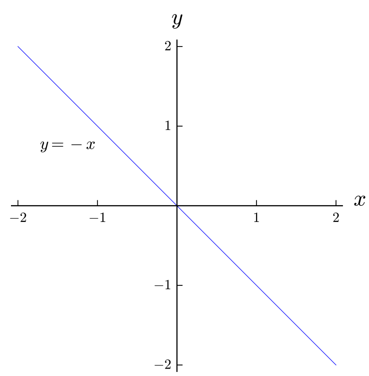
1. $y = c$



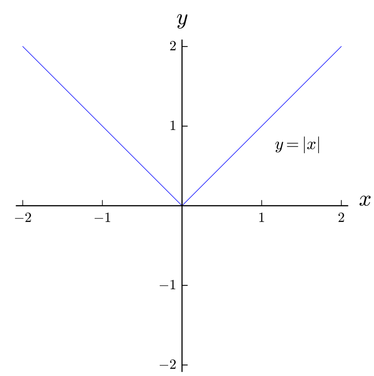
2. $y = x$



3. $y = -x$



4. $y = |x|$



New Graphs from Old

One of the main methods of graphing functions is to do so by modifying the graphs of familiar functions. There are a number of such modifications, which are summarized as follows.

New Graphs from Old

Let $f(x)$ be a function, and let c be a *positive* real number.

Function	Type of Modification
$y = f(x+c)$	shift the graph of $y = f(x)$ to the left by c units
$y = f(x-c)$	shift the graph of $y = f(x)$ to the right by c units
$y = f(x)+c$	shift the graph of $y = f(x)$ upward by c units
$y = f(x)-c$	shift the graph of $y = f(x)$ downward by c units
$y = cf(x)$	stretch the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$	stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$	reflect the graph of $y = f(x)$ in the x-axis
$y = f(-x)$	reflect the graph of $y = f(x)$ in the y-axis

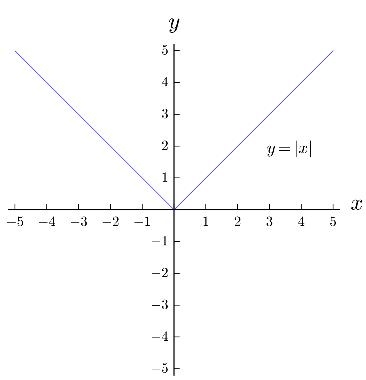
Of course, the various types of modifications listed above can be combined.

Example 1

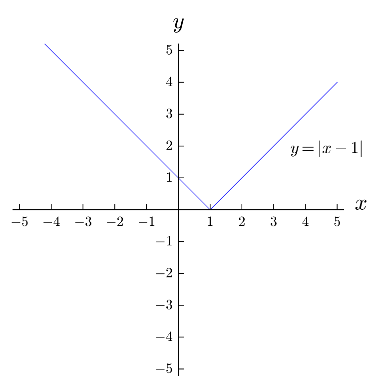
Sketch the graph of $y = -2|x-1| + 3$.

Solution As seen in the following figures, we sketch this graph in steps, starting with the graph of $y = |x|$, and then doing one modification of the graph at a time until we obtain the desired result.

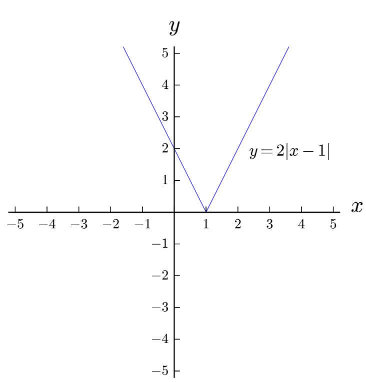
1. $y = |x|$



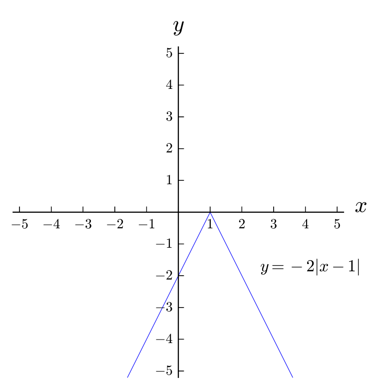
2. $y = |x-1|$



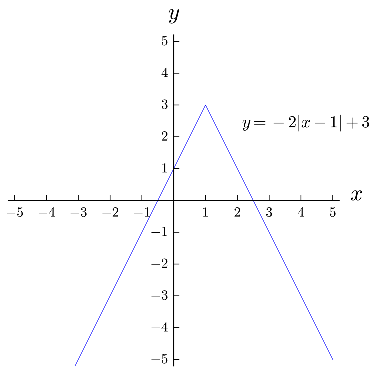
3. $y = 2|x-1|$



4. $y = -2|x-1|$



5. $y = -2|x-1|+3$



Functions Defined Piecewise

Whereas many commonly used functions are defined by a single formula, for example the function $f(x)$ defined by the formula $f(x) = x^2$, there are many functions that arise in mathematics and its applications that are defined in pieces, rather than by a single formula.

For example, let $g(x)$ be the function defined by the formula

$$g(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ x, & \text{if } x < 0. \end{cases}$$

This function $g(x)$ is defined for all real numbers x, and is well-defined, because there is a unique output for each input, due to the fact that the two parts of the function are defined on intervals that do not overlap.

The above function is and example of a function that is defined piecewise. Of course, a function that is defined piecewise can be defined in more than two parts; any number of parts is acceptable.

The graphs of functions that are defined piecewise are made by simply graphing each piece on the part of the real numbers for which it is defined. We use a solid dot to indicate the end of a piece of the graph where the graph is actually defined, and a hollow dot to indicate the end of a piece of the graph where the graph is not defined.

EXERCISES

1–4 ■ Find the domain of each function.

1. $f(x) = \dfrac{1}{x+2} - \dfrac{3}{x^2-9}$

2. $g(x) = \ln(x-3)$

3. $h(x) = \sqrt{x^2-16}$

4. $y = \dfrac{1}{\sqrt{x+3}}$

5–8 ■ Let $f(x)$ be the function defined by the formula $f(x) = x^2$. Find and simplify each of the following expressions.

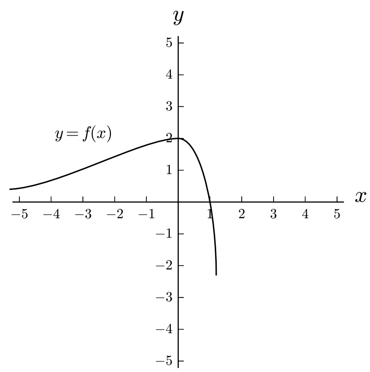
5. $f(x-2)$

6. $f(x+h) - f(x-h)$

7. $f(x+a+b)$

8. $\dfrac{f(x+h)-f(x)}{h}$

9–12 ■ Sketch the graph of each function, where the graph of $y = f(x)$ is seen below.



9. $y = f(x-3)$

10. $y = f(x)+2$

11. $y = -2f(x)$

12. $y = 2f(x+1)-3$

13–16 ■ Sketch the graph of each function.

13. $y = |x+2|$

14. $y = |x|+2$

15. $y = 2|x-3|$

16. $y = -|x|+1$

17–20 ■ Sketch the graph of each function.

17. $y = \begin{cases} x^2+1, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

18. $y = \begin{cases} |x-3|, & \text{if } x \geq 1 \\ 5x^2, & \text{if } x < 1. \end{cases}$

19. $y = \begin{cases} 3, & \text{if } x \geq 1 \\ x, & \text{if } -1 \leq x < 1 \\ -3, & \text{if } x < -1. \end{cases}$

20. $y = \begin{cases} \sin x, & \text{if } x \geq \frac{\pi}{2} \\ \tan x, & \text{if } -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ \cos x, & \text{if } x < -\frac{\pi}{2}. \end{cases}$

1.3 Linear Functions

Linear functions appear throughout mathematics and the application of mathematics in the sciences and social sciences. In particular, linear functions play a crucial role in calculus, because the derivative of a function is just the slope of the tangent line at each point of the graph of the function.

Linear functions are functions whose graphs are straight lines. It is assumed that you are familiar with straight lines from a geometric point of view.

Whereas straight lines exist in both the plane and three-dimensional space (and higher-dimensional space as well), at present we are concerned only with lines in the plane.

It is very important to distinguish between all lines in the plane on the one hand, and lines that are graphs of functions on the other hand. The difference between these two types of lines is that lines that are graphs of functions cannot be vertical.

Slope and y-intercept

A straight line in the plane that is the graph of a function has two significant numbers associated with it, namely, the slope and the y-intercept, which are defined as follows.

Slope and y-intercept

1. Let (x_0, y_0) and (x_1, y_1) be points. Suppose $x_0 \neq x_1$. The **slope** of the line containing (x_0, y_0) and (x_1, y_1) is $m = \dfrac{y_1 - y_0}{x_1 - x_0}$.
2. The **y-intercept** of a line is the value of y where the line intersects the y-axis.

The important thing to observe about the definition of the slope of a line is that no matter which two points on the line (x_0, y_0) and (x_1, y_1) are chosen, the ratio $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ will always be the same, which means that the slope of a line is well-defined. That would not be true for any curve other than a straight line.

The slope of a line measures how "slanted" the line is. For example, a slope of 0 means that the line is horizontal; a slope of 1 means that the line makes an angle of 45° with the positive x-axis; and a slope of -1 means that the line makes an angle of 45° with the negative x-axis.

Equation of a Line

Any line in the plane, whether vertical or not, can be given by an equation in x and y. There is a general form of the equation of a line that includes all lines, and then there are special forms for vertical lines and non-vertical lines.

Equation of a Line

1. The general form for the equation of a line is $ax + cy = d$.
2. The equation of a vertical line has the form $x = p$.
3. The equation of a non-vertical line has the form $y = mx + b$, where m is the slope of the line and

b is the y-intercept of the line.

Finding the Equation of a Line

From a geometric perspective, we need two points to determine a line. From an algebraic perspective, we need two pieces of information to determine the equation of a line; those two pieces of information can be the slope of the line and one point that is contained in the line, or they can be two points that are contained in the line.

Finding the Equation of a Line

1. Let m be a number and let (x_0, y_0) be a point. To find the equation of the line with slope m that contains (x_0, y_0), substitute (x_0, y_0) into the equation $y = mx + b$, solve for b, and substitute the result into $y = mx + b$.
2. Let (x_0, y_0) and (x_1, y_1) be points. Suppose $x_0 \neq x_1$. To find the equation of the line that contains (x_0, y_0) and (x_1, y_1), let $m = \dfrac{y_1 - y_0}{x_1 - x_0}$, substitute m into the equation $y = mx+b$, substitute (x_0, y_0) into the equation $y = mx + b$, solve for b, and substitute the result into $y = mx + b$.

Parallel and Perpendicular Lines

In the plane, two lines are parallel if they do not intersect, and they are not parallel if they do intersect. Intuitively, if two lines are parallel, then they "go in the same direction." More formally, we can use slope to determine if two lines are parallel, and also when two line are perpendicular.

Parallel and Perpendicular Lines

Let $y = mx + b$ and $y = nx + c$ be lines.

1. The two lines are parallel if and only if $m = n$.
2. The two lines are perpendicular if and only if $m = -\dfrac{1}{n}$.

Example 1

Find the equation of the line containing the point $(9, 11)$ that is perpendicular to the line $6x+10y = 7$.

Solution First, we solve for y in the equation of the given line, obtaining $y = -\frac{3}{5}x + \frac{7}{10}$. We deduce that the slope of this line is $n = -\frac{3}{5}$. The line whose equation we want to find is perpendicular to that line, and hence the slope of the line whose equation we want to find is $m = -\dfrac{1}{-\frac{3}{5}} = \frac{5}{3}$.

The equation we want to find has the form $y = mx + b$. Using the value of m that we found above, we see that this equation is $y = \frac{5}{3}x + b$. We now substitute $(x, y) = (9, 11)$ into that equation, which yields $11 = \frac{5}{3} \cdot 9 + b$. Solving for b we obtain $b = -4$. Hence the desired equation is $y = \frac{5}{3}x - 4$.

EXERCISES

1–4 ■ Find the equation of each line.

1. The line containing the point $(1, 2)$ that has slope 3

2. The line containing the point $(5, 0)$ that has slope -2

3. The line containing the point $(2, 1)$ that is parallel to the line $y = 5x - 3$

4. The line containing the point $(3, 2)$ that is perpendicular to the line $y = 2x + 1$

5–8 ■ Find the equation of the line containing each pair of points.

5. $(1, 2)$ and $(3, 8)$

6. $(2, 1)$ and $(-5, 6)$

7. $(3, 4)$ and $(-2, 4)$

8. $(4, 2)$ and $(4, -1)$

9–12 ■ For each pair of lines, state whether they are parallel, perpendicular or neither.

9. $y = 2x + 5$ and $6x - 3y = 4$

10. $y = 3x + 4$ and $x + 3y - 1 = 0$

11. $x - 4y + 2 = 0$ and $2x - 6y = 5$

12. $y = 7$ and $x = 2$

1.4 Polynomials

Linear functions are the simplest type of broadly useful functions, though of course not everything in the world is linear. The next simplest type of function is polynomial functions. Of course, all linear functions are polynomial functions, though not vice-versa.

Polynomial Functions

Some basic terminology about polynomial functions is the following.

Polynomial Functions

1. A **polynomial function** is a function $f(x)$ that can be defined by a formula of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers, and where $a_n \neq 0$.
2. The **coefficients** of the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ are the numbers $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$.
3. The **leading coefficient** of the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is the number a_n, which is never zero.
4. The **constant term** of the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is the number a_0.
5. The **degree** of the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is the number n.

The domain of any polynomial function is the set of all real numbers.

The range of every odd-degree polynomial function is the set of all real numbers (this fact is evident from the graphs of polynomials, though a proof is subtle).

The range of every even-degree polynomial function is not the whole set of real number. More precisely, if the leading coefficient of an even-degree polynomial is positive, then the polynomial has a minimum value, and the the range consists of all real numbers greater than or equal to this minimum value; if the leading coefficient of an even-degree polynomial is negative, then the polynomial has a maximum value, and the the range consists of all real numbers less than or equal to this maximum value.

Graphs of Polynomial Functions

In general, the graph of a polynomial function of degree n is a curve that intersects the x-axis at most n times, and has at most $n-1$ "bumps."

The graphs of polynomial functions do not have asymptotes (unless they are constant polynomials).

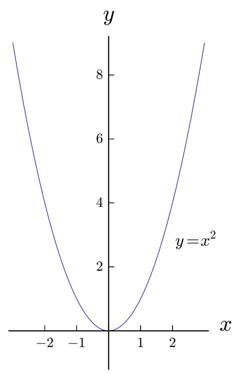
Graphs You Should Know

It is not possible to know the graph of every polynomial, but there are two that are used so often that they are worth knowing.

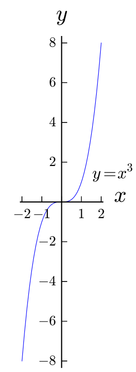
Graphs you should know: $y = x^2$ and $y = x^3$.

Polynomial Function Graphs

1. $y = x^2$



2. $y = x^3$



Roots of Polynomial Functions

In general, if a polynomial function is set equal to zero, there are at most n solutions (also called "roots" of the polynomial).

Every odd-degree polynomial function always has at least one root; even-degree polynomials might or might not have roots.

If $f(x)$ is a polynomial function, and if r is a root of $f(x)$, then $x - r$ is a factor of $f(x)$, which means that if $f(x)$ is divided by $x - r$, there will be no remainder.

Example 1

Solve the equation $x^3 + x^2 - 17x + 15 = 0$.

Solution Although there is a formula for solving cubic (third degree) equations, it is much more cumbersome to use than the quadratic formula, and in some cases, such as the equation we want to solve, there is an easier method, though it requires a bit of luck. A look at the coefficients of the polynomial $x^3 + x^2 - 17x + 15$ shows that the positive coefficients (which are 1, 1 and 15) add up to 17, and the single negative coefficient is -17. Hence, we guess that $x = 1$ is a root of the polynomial, which we verify by substituting it into the polynomial, which yields $1^3 + 1^2 - 17 \cdot 1 + 15$, which is indeed 0. We have therefore found one root of the polynomial.

From the above it follows that $x - 1$ is a factor of $x^3 + x^2 - 17x + 15$. By using long division, which is left to the reader, it can be seen that $x^3 + x^2 - 17x + 15 = (x-1)(x^2 + 2x - 15)$. We can then factor the quadratic polynomial, obtaining $x^3 + x^2 - 17x + 15 = (x-1)(x+5)(x-3)$.

Solving the original equation $x^3 + x^2 - 17x + 15 = 0$ is therefore equivalent to solving $(x-1)(x+5)(x-3) = 0$, and hence the solutions are $x = 1$, and $x = -5$ and $x = 3$.

EXERCISES

1–4 ■ Solve each equation.

1. $x^3 + 2x^2 - 3x = 0$

2. $x^3 - 2x^2 - x + 2 = 0$

3. $x^4 - 16 = 0$

4. $x^4 - 7x^2 + 10 = 0$

5–8 ■ Sketch the graph of each function.

5. $y = (x-3)^2$

6. $y = x^2 - 5$

7. $y = 2(x+1)^3$

8. $y = -4x^3 + 1$

1.5 Power Functions

Polynomials are made up of sums of expressions of the form x^2, x^3, x^4, x^5, etc., which are multiplied by coefficients. However, while raising x to a positive integer is a particularly simple way to raise x to a power, we can also raise x to numbers that are not positive integers, leading to the following type of function.

Power Functions

1. A **power function** is a function $f(x)$ that can be defined by a formula of the form $f(x) = x^a$, where a is a real number.
2. The **exponent** of the power function $f(x) = x^a$ is the number a.

It is important to stress that the exponent of a power function can be any real number, positive, negative or zero, and integer, fraction or irrational number. We consider each of these types of exponents.

Negative and Fractional Powers

We know what a power function is when the exponent is a positive integer. For example, we know that $x^3 = x \cdot x \cdot x$.

Power functions with exponents that are integers or fractions can be defined easily as follows.

Power Function Formulas

Let x be a real number, and let n, a and b be positive integers.

1. $x^0 = 1$.
2. $x^{-n} = \dfrac{1}{x^n}$.
3. $x^{\frac{1}{n}} = \sqrt[n]{x}$.
4. $x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$.

The formulas for defining power functions with exponents that are integers or fractions might seem arbitrary, but they make the various properties of exponential functions work out particularly nicely, as discussed in Section 1.7.

Irrational Powers

Whereas defining power functions with exponents that are integers or fractions is straightforward, it is not so simple—though still possible—to define power functions with exponents that are irrational numbers.

Suppose, for example, we want to define a function $f(x)$ by the formula $f(x) = x^\pi$. For a function to be meaningful, we need to be able to substitute numerical values for x and be able to obtain the numerical values of the function. But, for this function $f(x)$, what would $f(2)$ mean? That is, how would be compute 2^π? The answer is not at all obvious. Certainly, it seems reasonable to assume that because $3 < \pi < 4$, then 2^π should be between 2^3 and 2^4, meaning that $8 < 2^\pi < 16$. That's true, but not satisfactory.

A completely rigorous definition of 2^π requires more advanced mathematics than we have at our disposal here, but one approach is as follows. The number π is an irrational number, with decimal expansion that starts $3.14159\ldots$. Hence, we can approximate the value of π by the numbers 3, then 3.1, then 3.14, then 3.141, etc. We can rewrite each of these numbers as fractions, which are $\frac{3}{1}$, then $\frac{31}{10}$, then $\frac{314}{100}$, then $\frac{3141}{1000}$, etc. We know how to raise 2 to each of these exponents, which yields

$$
\begin{aligned}
2^3 &= 8 \\
2^{3.1} &= 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} = 8.5741\ldots \\
2^{3.1} &= 2^{\frac{314}{100}} = \sqrt[100]{2^{314}} = 8.8152\ldots \\
2^{3.1} &= 2^{\frac{3141}{1000}} = \sqrt[1000]{2^{3141}} = 8.8213\ldots \\
&\vdots \quad .
\end{aligned}
$$

If we keep doing this process, using more and more decimals of π, we will see that the output gets closer and closer to the number 8.8249…. Then we define $2^\pi = 8.8249\ldots$.

The above process makes use of the notion of a limit, which will be discussed in calculus. This process also seems arbitrary, in that the number π can be approached by a different sequence of fractions, but it can be proved rigorously that this method always yields the same number, no matter what sequence of fractions is used. A similar approach can be used to find the value of any number raised to an irrational exponent, and hence power functions with exponents that are irrational numbers are indeed defined.

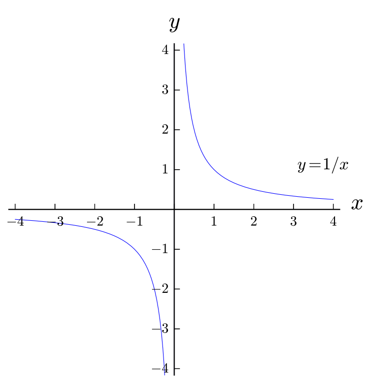
Graphs You Should Know

It is not possible to know the graph of every power function, but there are two that are used so often that they are worth knowing.

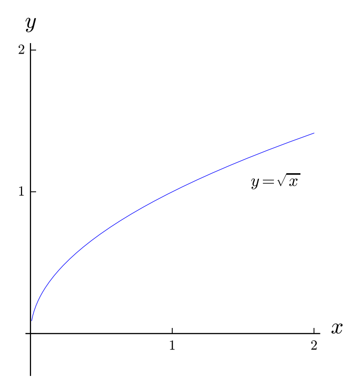
Graphs you should know: $y = \frac{1}{x}$ and $y = \sqrt{x}$.

Power Function Graphs

1. $y = \dfrac{1}{x}$



2. $y = \sqrt{x}$



EXERCISES

1–4 ■ Evaluate each expression without a calculator.

1. $4^{\frac{3}{2}}$ **2.** $27^{\frac{2}{3}}$

3. $16^{-\frac{1}{4}}$ **4.** $1000^{-\frac{4}{3}}$

5–8 ■ Rewrite each expression as a single power.

5. $\sqrt{a^5}$ **6.** $\dfrac{1}{\sqrt[3]{x^2}}$

7. $\sqrt{a}\,\sqrt[6]{b}$ **8.** $\dfrac{\sqrt{y}}{\sqrt[4]{x}}$

9–12 ■ Rewrite each expression as a single fraction and/or single root.

9. x^{-3} **10.** $b^{\frac{2}{5}}$

11. $y^{\frac{3}{2}}z^{-2}$ **12.** $\dfrac{a^{-4}}{b^{\frac{1}{3}}}$

13–16 ■ Sketch the graph of each function.

13. $y = \dfrac{1}{x+3}$ **14.** $y = -\dfrac{2}{x}$

15. $y = \sqrt{x-1} + 2$ **16.** $y = -3\sqrt{x} - 1$

1.6 Trigonometric Functions

When students first encounter trigonometry, it is usually in the context of the study of triangles. Whereas the study of triangles is very important in many parts of mathematics and its applications, for calculus our main interest in trigonometry is not the study of triangles, but is rather the six trigonometric functions, which arise from the study of triangles, but which are also useful in many other context, for example oscillatory motion.

Radians and Degrees

The study of triangles involves the measurement of angles. As with other types of measurements, for example length, volume and weight, the measurement of angles involves units of measurement. And, just as there are various units that are used for the measurement of length (for examples, inches and centimeters), so too for angles there are various units that can be used. The units for measuring angles that is the most commonly used in elementary school, middle school and high school are degrees. For calculus, however, it is important to stress that degrees are the wrong units to be used for measuring angles, and instead the only units that should be used for calculus are radians.

The problem with degrees is that they are a completely arbitrary unit of measurement. One degree is obtained by taking a complete angle around a point and dividing it into 360 equal parts. The choice of 360 is completely arbitrary; any other number could have been used. By contrast, radians are not arbitrary, because they are based upon the circumference of the unit circle (that is, the circle of radius 1 centered at the origin in the plane).

Because we all learn about degrees before learning about radians, the key to using radians is to know how to convert degrees to radians and vice-versa. The key to that conversion is to recall that the circumference of the unit circle is 2π, and the unit circle corresponds to a complete angle around a point, which is $360°$. Hence $360°$ is equal to 2π radians, which leads to the following method for conversion between degrees and radians.

Degrees to Radians Conversion

1. To convert an angle in degrees to radians, multiply by $\frac{\pi}{180}$.
2. To convert an angle in radians to degrees, multiply by $\frac{180}{\pi}$.

Some regularly used conversions between degrees and radians are the following.

Degrees to Radians Conversion: Standard Angles

1. $90° = \frac{\pi}{2}$ rad.
2. $180° = \pi$ rad.
3. $270° = \frac{3\pi}{2}$ rad.
4. $360° = 2\pi$ rad.

The Six Trigonometric Functions

When you first learn about the six trigonometric functions, it is usually in the context of angles in right triangles. That approach is correct, but limited in use, because the angles in a right triangle must be between 0° and 90°, or, stated properly in radians, between 0 and $\frac{\pi}{2}$. For calculus and its applications, by contrast, we need the six trigonometric functions defined for all real numbers (in the case of sine and cosine), and almost all real numbers (in the case of tangent, cotangent, secant and cosecant).

In order to define the six trigonometric functions for all (or almost all) real numbers, we use the unit circle, which has equation $x^2 + y^2 = 1$. Specifically, let t be a real number, which we think of as an angle measured in radians. We plot this angle at the origin, starting with the positive x-axis and going counterclockwise, which then gives rise to a ray starting at the origin. This ray intersects the unit circle at a point (x, y). We then form a right triangle, with one side of the triangle the line segment going from (x, y) to $(x, 0)$, with the other side of the triangle the line segment going from $(x, 0)$ to $(0, 0)$, and with the hypotenuse the radius of the unit circle going from $(0, 0)$ to (x, y). See Figure 2 of this section for what happens when (x, y) is in the first quadrant; similar figures occur when (x, y) is in the other quadrants.

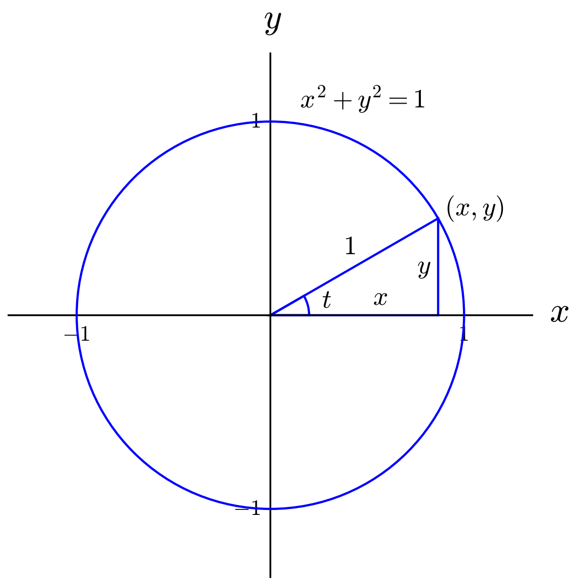


Figure 2: Unit Circle and Trigonometric Functions

Using the point (x, y) on the unit circle, we then define the six trigonometric functions of t as follows:

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}$$
$$\sec t = \frac{1}{x} \qquad \csc t = \frac{1}{y} \qquad \cot t = \frac{x}{y}.$$

We note that $\sin t$ and $\cos t$ are defined for all real numbers t. By contrast, we see that $\tan t$ and $\sec t$ are defined whenever $x \neq 0$, which is when x is not any of $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. Similarly, we see that $\cot t$ and $\csc t$ are defined whenever $y \neq 0$, which is when x is not any of $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$.

We also note that if t is between 0 and $\frac{\pi}{2}$, then the above definition of the six trigonometric functions of t is the same as the definition given for angles in a Sright triangle, because the hypotenuse in the right triangle has length 1. For example $\sin t$ is "opposite over hypotenuse," and similarly for the other trigonometric functions.

Trigonometric Functions of Standard Angles

For most angles, a calculator or computer is needed to calculate the various trigonometric functions of that angle. (When you are using radians to measure angles, and you are using a calculator, made sure the calculator is set for radians rather than degrees.) However, there a few angle that occur so frequently that it is worth knowing the values of sine and cosine of these angle. These values are as follows.

Sine of Standard Angles

1. $\sin 0 = 0$.
2. $\sin \frac{\pi}{6} = \sin 30° = \frac{1}{2}$.
3. $\sin \frac{\pi}{4} = \sin 45° = \frac{\sqrt{2}}{2}$.
4. $\sin \frac{\pi}{3} = \sin 60° = \frac{\sqrt{3}}{2}$.
5. $\sin \frac{\pi}{2} = \sin 90° = 1$.

Cosine of Standard Angles

1. $\cos 0 = 1$.
2. $\cos \frac{\pi}{6} = \cos 30° = \frac{\sqrt{3}}{2}$.
3. $\cos \frac{\pi}{4} = \cos 45° = \frac{\sqrt{2}}{2}$.
4. $\cos \frac{\pi}{3} = \cos 60° = \frac{1}{2}$.
5. $\cos \frac{\pi}{2} = \cos 90° = 0$.

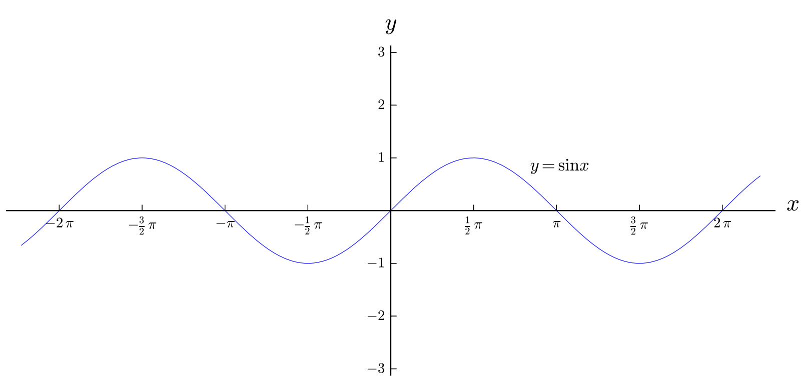
Observe that the values listed in Items (1)–(5) in the above chart for the values of cosine are in backwards order from the values listed in Items (1)–(5) in the above chart for the values of sine; that symmetry makes these values easier to remember. One way of remembering these five values is that for sine, the five values in order are $\dfrac{\sqrt{0}}{2}, \dfrac{\sqrt{1}}{2}, \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{3}}{2}$ and $\dfrac{\sqrt{4}}{2}$, which has a simple pattern.

Graphs of Functions

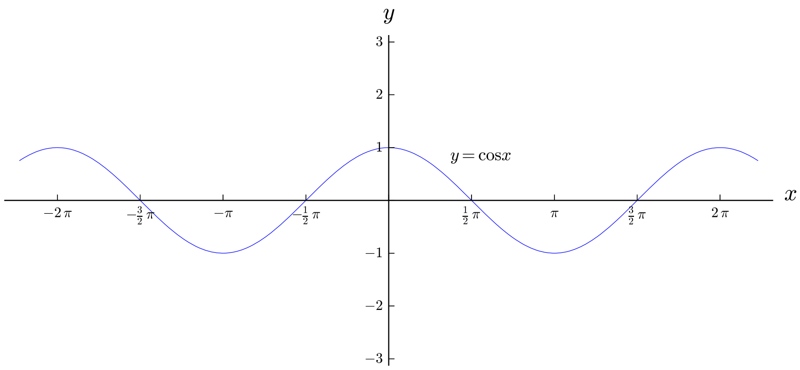
Graphs you should know: $y = \sin x$, and $y = \cos x$ and $y = \tan x$.

Basic Trigonometric Function Graphs

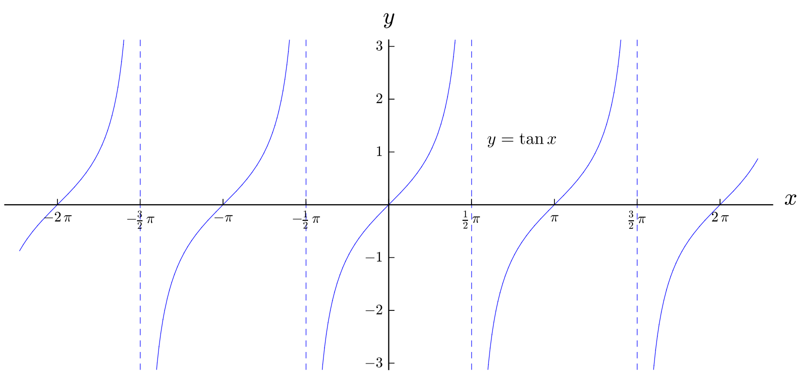
1. $y = \sin x$



2. $y = \cos x$



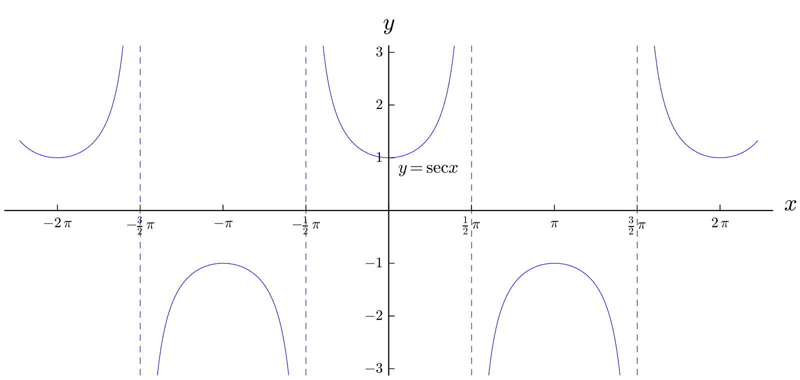
3. $y = \tan x$



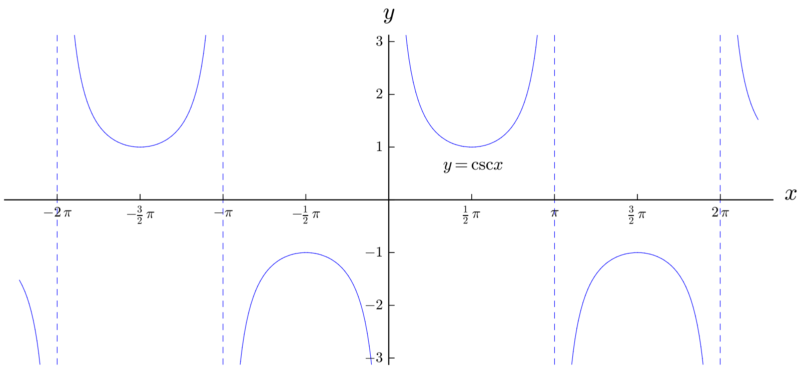
It is worth seeing (though not necessarily memorizing) the graphs of the other three trigonometric functions: $y = \sec x$, and $y = \csc x$ and $y = \cot x$.

Optional: Additional Trigonometric Function Graphs

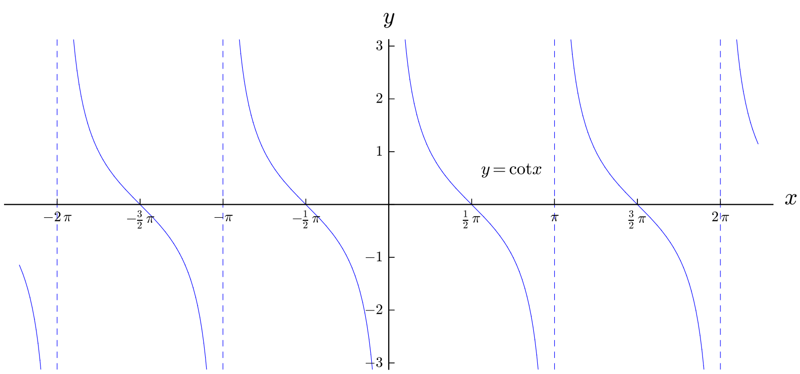
1. $y = \sec x$



2. $y = \csc x$



3. $y = \cot x$



Trigonometric Identities

There are a number of relations, called trigonometric identities, between various of the six trigonometric functions. The most basic trigonometric identities are the following.

Basic Trigonometric Formulas

Let x be a real number.

1. $\tan x = \frac{\sin x}{\cos x}$.
2. $\sec x = \frac{1}{\cos x}$.
3. $\csc x = \frac{1}{\sin x}$.
4. $\cot x = \frac{\cos x}{\sin x}$.
5. $\sin(x + 2\pi) = \sin x$.
6. $\cos(x + 2\pi) = \cos x$.
7. $\sin^2 x + \cos^2 x = 1$.

There are a a number of other useful trigonometric identities that are useful in calculus on occasion. It is not necessary to memorize these formulas, but it is important to know that they exist, and to be able to find them when needed.

Optional: Additional Trigonometric Formulas

Let x and y be real numbers.

1. $\sin(-x) = -\sin x$.
2. $\cos(-x) = \cos x$.
3. $\sin(x+y) = \sin x \cos y + \cos x \sin y$.
4. $\sin(x-y) = \sin x \cos y - \cos x \sin y$.
5. $\cos(x+y) = \cos x \cos y - \sin x \sin y$.
6. $\cos(x-y) = \cos x \cos y + \sin x \sin y$.
7. $\sin(2x) = 2\sin x \cos x$.
8. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.
9. $\sin^2 x = \dfrac{1-\cos(2x)}{2}$.
10. $\cos^2 x = \dfrac{1+\cos(2x)}{2}$.

EXERCISES

1–4 ■ Convert each of the following angle given in degrees to radians without calculator.

1. $45°$ **2.** $-30°$

3. $135°$ **4.** $330°$

5–8 ■ Convert each of the following angle given in radians to degrees without calculator.

5. 4π **6.** $\dfrac{\pi}{6}$

7. $-\dfrac{3\pi}{4}$ **8.** $\dfrac{5\pi}{12}$

9–12 ■ Evaluate each expression without a calculator.

9. $\sin\left(\dfrac{5\pi}{2}\right)$ **10.** $\cos\left(-\dfrac{\pi}{6}\right)$

11. $\tan\left(\dfrac{\pi}{3}\right)$ **12.** $\csc\left(\dfrac{\pi}{2}\right)$

13–20 ■ Suppose that angles α and β are between 0 and $\dfrac{\pi}{2}$, and that $\sin\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{12}{13}$. Evaluate each expression without a calculator.

13. $\sin(-\alpha)$ **14.** $\cos(-\beta)$

15. $\cos\alpha$ **16.** $\sin\beta$

17. $\sin(2\alpha)$ **18.** $\cos(2\alpha)$

19. $\sin(\alpha+\beta)$ **20.** $\cos(\alpha+\beta)$

21–24 ■ Sketch the graph of each function.

21. $y = \sin x + 3$

22. $y = \tan(x - \frac{\pi}{2})$

23. $y = 3\sec x$

24. $y = \cos(2x)$

1.7 Exponential Functions

In Section 1.5 we saw power functions, which are functions defined by formulas of the form $f(x) = x^a$ for some real number a. By contrast, we now discuss exponential functions, which are as follows.

Exponential Functions

1. An **exponential function** is a function $f(x)$ that can be defined by a formula of the form $f(x) = a^x$, where a is a positive real number.
2. The **base** of the exponential function $f(x) = a^x$ is the number a.

Consider the function $f(x)$ given by the formula $f(x) = 2^x$. We know what $f(3)$ means, because $f(x) = 2^3 = 2 \cdot 2 \cdot 2 = 8$. We also know what $f(-5)$ and $f(\frac{4}{7})$ means, because we know how to raise the number 2 to negative and fractional powers, as discussed in Section 1.5. What would $f(\pi)$ mean? It equals 2^π, and, although it isn't easy to compute that by hand, we already saw what that means when we discussed irrational powers in Section 1.5. Although the discussion in Section 1.5 was for power functions rather than exponential functions, there is no difference when evaluating specific numbers such as 2^π, because the same number arises when we substitute $x = \pi$ into the function $f(x)$ defined by the formula $f(x) = 2^x$ and when we substitute the number $x = 2$ into the function $g(x)$ defined by the formula $g(x) = x^\pi$.

☞ **Error Warning** Although the same consideration arises when dealing with irrational exponents in power functions and exponential functions, it is very important not to confuse these two types of functions, because power functions and exponential functions are very different from each other, with different graphs, different behaviors and different uses.

Among the various differences between power functions and exponential functions is the fact that exponential functions grow much faster than polynomial functions as x gets larger. For example, let $f(x)$ be the function defined by the formula $f(x) = 2^x$ and let $g(x)$ be the function defined by the formula $g(x) = x^2$. Compute each of $f(10)$ and $g(10)$, and compare the values.

Exponential functions are extremely useful in many aspects of mathematics and its applications, for example population growth, radioactive decay, differential equations, compound interest, and more.

The Number e and the Function e^x

It is possible to have an exponential function defined by a formula of the form $f(x) = a^x$ for any positive real number a. It turns out, however, that there is one particular number that yields the most useful exponential function, and that is the number e, defined as follows.

The Number e

1. The number $\boldsymbol{e}$ is defined by the formula

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

2. The numerical value of e is $e = 2.718\ldots$.

The above definition of the number e might seem unmotivated, but the number e turns out to be extremely important in mathematics and its applications. Indeed, the number e is of similar importance as the number π (and, in fact, there is a relation between these two numbers).

Though it is not possible to explain it right now, the function $f(x) = e^x$ works more nicely than exponential functions with other bases. You will see why $f(x) = e^x$ is better when you learn about the derivatives of exponential functions in calculus.

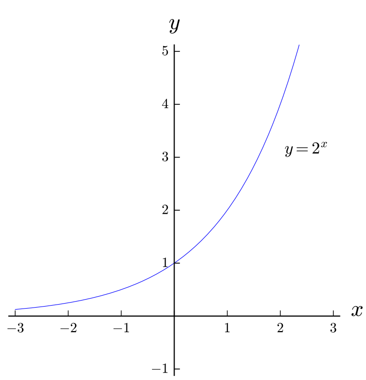
Graphs of Exponential Functions

The graphs of all exponential functions have the same shape, and are just small variations of each other. Specifically, when $a > 1$, all graphs of the form $y = a^x$ are increasing, where the graph increases faster if a is larger; when $0 < a < 1$, all graphs of the form $y = a^x$ are decreasing, where the graph decreases faster if a is smaller; and $y = 1^x$ is just the constant function $y = 1$.

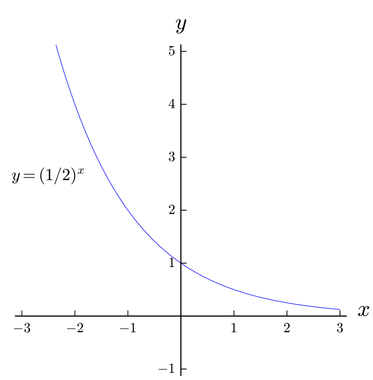
Graphs you should know: $y = a^x$ when $a > 1$, and $y = a^x$ when $0 < a < 1$, and $y = e^x$.

Exponential Function Graphs

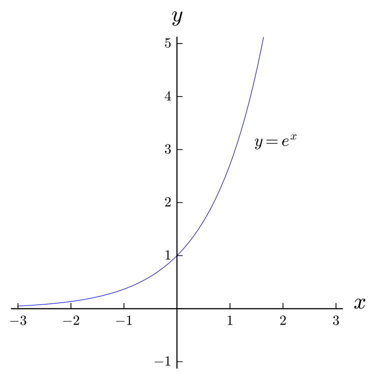
1. $y = a^x$, when $a > 1$



2. $y = a^x$, when $0 < a < 1$



3. $y = e^x$



Properties of Exponentials

Exponential functions have a number of very nice properties, including the following.

Exponential Function Formulas

Let a be a positive real number, and let x and y be real numbers.

1. $a^{x+y} = a^x a^y$.
2. $a^{x-y} = \frac{a^x}{a^y}$.
3. $(a^x)^y = a^{xy}$.
4. $a^0 = 1$.
5. $(ab)^x = a^x b^x$.
6. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

EXERCISES

1–4 ■ Simplify the following expressions. Each answer should contain only one occurrence of each letter, and should have only positive exponents.

1. $3x^{-2}y^5 \cdot (2x^2y)^3$

2. $(ab^2c^3)^{-3} \cdot (2a^{-3}bc^2)^2$

3. $\dfrac{3s^4t^{-5}}{6s^{-2}t^{-4}}$

4. $\left(\dfrac{m^{-2}n^5p^2}{m^4n^3p^{-2}}\right)^3$

5–8 ■ Sketch the graph of each function.

5. $y = e^{x-2}$

6. $y = e^x - 1$

7. $y = 2e^{x+3}$

8. $y = -e^x - 4$

1.8 Logarithmic Functions

Though some people find logarithmic functions to be mysterious, due to the somewhat indirect way in which they are usually defined, they are in fact just normal functions, which happen to very useful.

Another misconception about logarithms is related to their history. They originally arose in order to simplify lengthy numerical calculations when such calculations where done by hand (which was most of human history), and, the thinking goes, now that we have calculators and computers, perhaps we should not be interested in logarithms any more. In fact, a mathematical method can arise for one reason, but then turn out to be useful for other reasons, even when the original reason is no longer relevant (the same holds for some non-mathematical issues); logarithms are an example of that phenomenon. Not only are logarithms useful in a variety of places in mathematics and its applications, but, given the relation between exponential functions and logarithmic functions, any place where exponential functions are used, logarithms are usually needed too.

There are a number of ways to think of how to define logarithms. One way, to be discussed in calculus, in via the notion of inverse function; in particular, logarithmic functions are simply the inverse functions of exponential functions. For now we use the following equivalent approach.

Logarithmic Functions

1. Let a and x be positive real numbers. Suppose that $a \neq 1$. The expression $\log_a x$ is defined to be the number y such that $a^y = x$.
2. A **logarithmic function** is a function $f(x)$ that can be defined by a formula of the form $f(x) = \log_a x$, where a is a positive real number such that $a \neq 1$.
3. The **base** of the logarithmic function $f(x) = \log_a x$ is the number a.

Common and Natural Logarithmic Functions

Just as $f(x) = e^x$ is the most important exponential functions, so too the logarithmic function with base e, denoted as follows, is the most important logarithmic function.

Natural Logarithmic Function

1. The **natural logarithmic function** is the function $f(x) = \ln x$, where $\ln x$ is an abbreviation for $\log_e x$.
2. The **base** of the natural logarithmic function $f(x) = \ln x$ is the number e.

Another logarithmic function with its own name is the "common logarithm," which is the logarithmic function that has base 10. The common logarithmic function is often denoted $\log x$. Whereas the common logarithm was very important when logarithms were used to help do numerical calculations, the common logarithm is not as important to us today.

Graphs of Logarithmic Functions

If one takes the approach to logarithmic functions of viewing them as the inverse functions of exponential functions, then it follows that the graphs of logarithmic functions are obtained by reflecting the graphs of the corresponding exponential functions in the line $y = x$.

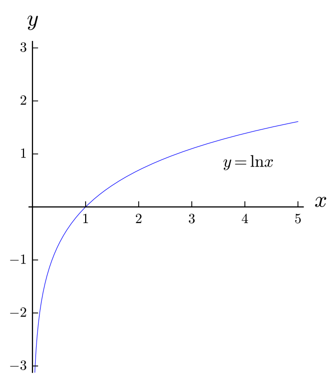
It follows that the graphs of all logarithmic functions have the same shape, and are just small variations of each other. Specifically, when $a > 1$, all graphs of the form $y = \log_a x$ are increasing, where the graph increases more slowly if a is larger; when $0 < a < 1$, all graphs of the form $y = a^x$ are decreasing, where the graph decreases more slowly if a is smaller.

Because we will mostly use the natural logarithmic function, its graph is the important one.

Graphs you should know: $y = \ln x$.

Logarithm Function Graphs

1. $y = \ln x$



Properties of Logarithms

Logarithmic functions have a number of very nice properties, including the following. We state these properties first in general, and then in the specific case of the natural logarithmic function (which is redundant, but is so important that it is worth stating explicitly).

Logarithm Function Formulas

Let a, x and y positive real numbers.

1. $\log_a(xy) = \log_a x + \log_a y$.

2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

3. $\log_a x^r = r \log_a x$.

4. $\log_a 1 = 0$.

5. $\log_a(a^x) = x$.

6. $a^{\log_a x} = x$.

7. $\log_a x = y \;\Leftrightarrow\; a^y = x$.

Natural Logarithm Function Formulas

Let a, x and y positive real numbers.

1. $\ln(xy) = \ln x + \ln y$.
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
3. $\ln x^r = r \ln x$.
4. $\ln 1 = 0$.
5. $\ln(e^x) = x$.
6. $e^{\ln x} = x$.
7. $\ln x = y \Leftrightarrow e^y = x$.

EXERCISES

1–4 ■ Simplify each expression.

1. $\log_3 9$ **2.** $\log_9 3$

3. $\log_2 \frac{1}{32}$ **4.** $\log_2 8$

5–8 ■ Rewrite each expression as a single logarithm.

5. $\log_5 p^4 - \log_5 p$ **6.** $3\ln x + 4\ln x$

7. $5\log_2 x - 3\log_2 z$ **8.** $\frac{1}{2}\ln a - \frac{1}{3}\ln b$

9–12 ■ Rewrite each expression in terms of $\ln x$, $\ln y$ and $\ln z$.

9. $\ln \sqrt[5]{x}$ **10.** $\ln(x^2y^5z)$

11. $\ln\left(\frac{x^3}{z^2}\right)$ **12.** $\ln\sqrt{x^4y^3}$

13–16 ■ Simplify each expression.

13. $e^{\ln(2a+b)}$ **14.** $\ln(e^{xy})$

15. $e^{2\ln x + 3\ln w}$ **16.** $\ln(e^{m^2}e^{3n})$

17–20 ■ Solve each equation.

17. $e^{x+5} = 3$ **18.** $\ln(x^2 - 3) = 4$

19. $e^{x+2} + e^x = 8$ **20.** $\ln(x+1) - \ln x = 1$

21–24 ■ Sketch the graph of each function.

21. $y = \ln(x+1)$ **22.** $y = \ln x + 1$

23. $y = 2\ln(x-3)$ **24.** $y = -\ln x + 2$