Homework Solutions: Proofs & Fundamentals

$\boxed{\text{FACT: } x \in f^*(T) \iff f(x) \in T}$

$\boxed{4.2.12}$: Let $f: A \to B$ and $C, D \subseteq B$.
Then $f^*(D-C) = f^*(D) - f^*(C)$.

proof:

$\subseteq$: Let $x \in f^*(D-C)$. This means $f(x) \in D-C$. Thus $f(x) \in D$ and $f \notin C$. Since $f(x) \in D$, we have $x \in f^*(D)$, and since $f(x) \notin C$, we have $x \notin f^*(C)$. (using the FACT above) Hence $x \in f^*(D) - f^*(C)$, so LHS $\subseteq$ RHS.

$\supseteq$: Let $x \in f^*(D) - f^*(C)$. Then $x \in f^*(D)$ and $x \notin f^*(C)$. Thus, by the Fact, $f(x) \in D$ and $f(x) \notin C$, so $f(x) \in D - C$. By the Fact again, $x \in f^*(D-C)$, so RHS $\subseteq$ LHS.

$\boxed{4.2.10}$: (1) Here's one example: Let $A = \{1,2\}$, $B = \{3\}$. Let $f: A \to B$ be defined by $f(1) = 3$, $f(2) = 3$. Let $P = \{1\}$, $Q = \{1,2\}$.
Then $f_*(P) = \{3\} = f_*(Q)$, but $\begin{cases} P \subseteq Q \\ P \neq Q \end{cases}$.

(2) Let $C = \{1\}$, Let $D = \{2,3\}$. Let $g: C \to D$ be defined by $g(1) = 2$. Let $S = \{2\}$, and $T = \{2,3\}$. Then

$$f^*(S) = \{1\} = f^*(T),$$

but $S \neq T$, and $S \subseteq T$.

There are, of course, many other correct examples.