Applying number theory and algebraic geometry to cryptography

Alice Silverberg

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AMS Meeting, Bard College

Goal

We will discuss some ways to use algebraic tori, elliptic curves, and abelian varieties to

- better understand some known cryptosystems,
- make them more efficient,
- create new efficient cryptosystems,
- attack the security of known cryptosystems.

Talk Outline

- Discrete Log Based Cryptography (Diffie-Hellman, El Gamal)
- Improvements (LUC, XTR, $\mathbb{T}_2$, CEILIDH, and beyond)
- Mathematical Interpretation via Algebraic Tori
- Elliptic Curve Analogue

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$$|\mathbb{F}_q^\times| = q - 1$$

The multiplicative group $\mathbb{F}_q^\times$ is cyclic.

$$\mathbb{F}_q^\times = \{g, g^2, g^3, \ldots, g^{q-1} = 1\}$$

for some $g \in \mathbb{F}_q^\times$.

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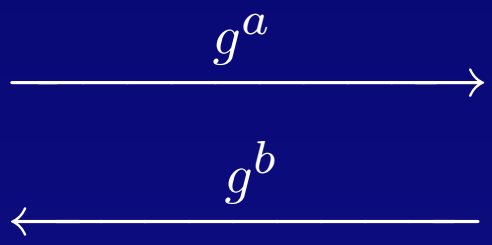

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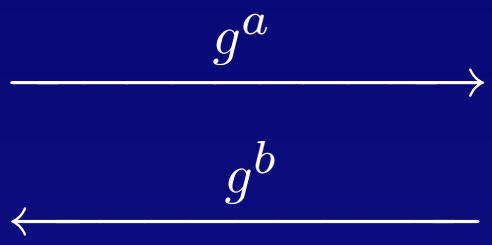

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Diffie-Hellman Problem:

Given g, g^a, and g^b, find g^{ab}.

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Diffie-Hellman Problem:

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Discrete Logarithm Problem:

Given g and g^a, find a.

Cryptographic Security

For a general cyclic group of order q, there are attacks on the Discrete Logarithm Problem that run in time exponential in $\log q$.

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The goal is to find groups of a given size (suitable for cryptography) for which the smallest $\mathbb{F}_q^\times$ they embed in is large. (Also need the group order to have a large prime divisor.)

Our Finite Field Goals

Use $\mathbb{F}_{q^n}^{\times}$ to improve the efficiency by a factor of $n/\varphi(n)$, where φ is Euler's phi function.

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Want $n/\varphi(n)$ large. Optimal n: $1, 2, 6, 30, \ldots$

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$\mathbb{F}_q \subset \mathbb{F}_{q^2}$ $\qquad$ $|\mathbb{F}_{q^2}^\times| = q^2 - 1 = (q+1)(q-1)$

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They share $\mathrm{Tr}(g^{ab})$. (Exercise!)

Lucas-based Key Agreement

The security is based on the difficulty of the Diffie-Hellman Problem in $\mathbb{F}_{q^2}^{\times}$.

Each party transmits only one element of $\mathbb{F}_q$ (rather than one element of $\mathbb{F}_{q^2}$).

This doubles the efficiency.

W. Müller & W. Nöbauer (1981)
H. C. Williams (1985)
P. Smith (1993): **LUC**

XTR (2000)

$$|\mathbb{F}_{q^6}^{\times}| = q^6 - 1 = (q^2 - q + 1)(q^2 + q + 1)(q^2 - 1)$$

Take $g \in \mathbb{F}_{q^6}^{\times}$ of order $q^2 - q + 1$.

$$\mathrm{Tr} : \mathbb{F}_{q^6} \longrightarrow \mathbb{F}_{q^2}, \qquad x \mapsto x + x^{q^2} + x^{q^4}$$

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XTR (2000)

The security is based on the difficulty of the Diffie-Hellman Problem in $\mathbb{F}_{q^6}^{\times}$ (note that g does not lie in any proper subfield of $\mathbb{F}_{q^6}$).

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This triples the efficiency.

A. Brouwer, R. Pellikaan, & E. Verheul (1999)
A. Lenstra & E. Verheul (2000)

Gong-Harn Key Agreement (1999)

$|\mathbb{F}_{q^3}^{\times}| = q^3 - 1 = (q^2 + q + 1)(q - 1)$

Take $g \in \mathbb{F}_{q^3}^{\times}$ of order $q^2 + q + 1$.

$\mathrm{Tr} : \mathbb{F}_{q^3} \longrightarrow \mathbb{F}_q, \quad x \mapsto x + x^q + x^{q^2}$

$\gamma : \mathbb{F}_{q^3} \longrightarrow \mathbb{F}_q, \quad x \mapsto x \cdot x^q + x \cdot x^{q^2} + x^q \cdot x^{q^2}$

(2 of the 3 symmetric functions on $\{x, x^q, x^{q^2}\}$)

Gong-Harn Key Agreement (1999)

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		$\xleftarrow{\mathrm{Tr}(g^b),\gamma(g^b)}$	

They share $\mathrm{Tr}(g^{ab})$ and $\gamma(g^{ab})$. (Exercise)

The security is based on the difficulty of the Diffie-Hellman Problem in $\mathbb{F}_{q^3}^{\times}$.

Each party transmits only 2 element of $\mathbb{F}_q$.

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Alice's private key: a
Alice's public key: g^a

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The CEILIDH cryptosystem can multiply and exponentiate.

XTR

XTR = **ECSTR** =

Efficient Compact Subgroup Trace Representation

Silverberg, AMS Bard

CEILIDH

CEILIDH is Compact, Efficient, Improves on LUC, and Improves on Diffie-Hellman

Silverberg, AMS Bard

CEILIDH can multiply; LUC and XTR can't

Silverberg, AMS Bard

CEILIDH (2003, Rubin & Silverberg)

As in XTR, take $g \in \mathbb{F}_{q^6}^{\times}$ of order $q^2 - q + 1$.

Let

$$G_6 = \{g, g^2, \ldots, g^{q^2-q+1} = 1\}.$$

CEILIDH (2003, Rubin & Silverberg)

We found an efficiently computable "compression" function f, satisfying the following properties:

- $f : G_6 \dashrightarrow \mathbb{F}_q \times \mathbb{F}_q$ is defined almost everywhere,
- can compute $f(g^{ab})$ from $f(g^a)$ and b,
- can compute $f(hk)$ from $f(h)$ and $f(k)$,
- f has an efficiently computable inverse

$$j : \mathbb{F}_q \times \mathbb{F}_q \to G_6.$$

CEILIDH

Use $f : G_6 \dashrightarrow \mathbb{F}_q \times \mathbb{F}_q$ to compress powers of $g \in \mathbb{F}_{q^6}^{\times}$.

Use $j : \mathbb{F}_q \times \mathbb{F}_q \longrightarrow G_6$ to decompress elements of $\mathbb{F}_q \times \mathbb{F}_q$, in order to use the group law in $G_6 \subset \mathbb{F}_{q^6}^{\times}$.

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CEILIDH El Gamal Encryption

Alice's private key: a
Alice's public key: $f(g^a) \in \mathbb{F}_q^2$

- Bob represents the message M as a power of g, and picks a random r. The ciphertext is

$$(f(g^r), f(M \cdot j(f(g^a))^r)).$$

- To decrypt a ciphertext (c, d), Alice computes $M = j(d) \cdot j(c)^{-a}$.

The $\mathbb{T}_2$ cryptosystem

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As was the case for the Lucas-based cryptosystems, take $g \in \mathbb{F}_{q^2}^{\times}$ of order $q+1$.

Let

$$G_2 = \{g, g^2, \ldots, g^{q+1} = 1\}.$$

Write

$$\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{D}).$$

The $\mathbb{T}_2$ cryptosystem

$$G_2 - \{1, -1\} \xrightarrow{f} \mathbb{F}_q$$

$$a + b\sqrt{D} \xmapsto{f} \frac{1+a}{b}$$

$$G_2 \xleftarrow{j} \mathbb{F}_q$$

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To do Diffie-Hellman and El Gamal, use the group law on G_2, and use f to efficiently represent the elements of G_2 in $\mathbb{F}_q$.

The $\mathbb{T}_2$ cryptosystem

There's no need to decompress elements of $\mathbb{F}_q$, since one can use

$$j(\alpha)j(\beta) = j\left(\frac{\alpha\beta + D}{\alpha + \beta}\right)$$

to translate G_2's multiplication to $\mathbb{F}_q$.

Can do everything in $\mathbb{F}_q$, without using $\mathbb{F}_{q^2}$.

An example of CEILIDH

Choose a prime power $q \equiv 2$ or $5 \pmod 9$.

Let $\zeta_9 \in \mathbb{F}_{q^6}$ be a primitive ninth root of 1 and let $\zeta_3 = \zeta_9^3$.

We'll define

$$G_6 - \{1, \zeta_3^2\} \xrightarrow{f} \mathbb{F}_q \times \mathbb{F}_q$$

$$G_6 \xleftarrow{j} \mathbb{F}_q \times \mathbb{F}_q$$

An example of CEILIDH

$$a + b\zeta_3 \xrightarrow{\ f\ } \left(\frac{u}{w}, \frac{v}{w}\right)$$

$$\frac{s+r\zeta_3}{s+r\zeta_3^2} \xleftarrow{\ j\ } (x, y)$$

where $a, b \in \mathbb{F}_{q^3}$,

$$\frac{1+a}{b} = w + u(\zeta_9 + \zeta_9^{-1}) + v(\zeta_9^2 + \zeta_9^{-2}),$$

$$r = 1 - x^2 - y^2 + xy,$$

$$s = 1 + x(\zeta_9 + \zeta_9^{-1}) + y(\zeta_9^2 + \zeta_9^{-2}).$$

Beyond XTR

A. Lenstra and others asked if one can use symmetric polynomials to generalize XTR to $\mathbb{F}_{q^{30}}$ and beyond.

In "Looking beyond XTR", W. Bosma, J. Hutton, & E. Verheul (2002) gave some precise conjectures for 30 and beyond, involving symmetric polynomials.

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K. Rubin and I disproved the conjectures and showed that symmetric functions are not the right functions to consider.

Cyclotomic Polynomials

Let Φ_n denote the n-th cyclotomic polynomial:

$$\Phi_1(x) = x - 1,$$

$$\Phi_2(x) = x + 1,$$

$$\Phi_6(x) = x^2 - x + 1,$$

$$\deg(\Phi_n(x)) = \varphi(n),$$

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \quad \Rightarrow \quad \Phi_n(q) | (q^n - 1) = |\mathbb{F}_{q^n}^{\times}|$$

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The next interesting case is $n = 30$; here $n/\varphi(n) = 30/8 = 3.75$.

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M. van Dijk and D. Woodruff (2004) found a clever way to use the "excess" $\mathbb{F}_q^A$ to encode the message being encrypted or signed, so that it doesn't go to waste.

van Dijk-Woodruff's Asymptotically Optimal Cryptosystems

They use

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)},$$

where $\mu(m) = (-1)^r$ if m is a product of r distinct primes, to obtain an "almost bijection" between $G_n \times \mathbb{F}_q^A$ and $\mathbb{F}_q^B$, with $A = \sum_{d|n, \mu(n/d)=-1} d$ and $B = \sum_{d|n, \mu(n/d)=1} d$.

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This gives an "almost bijection" between $G_{30} \times \mathbb{F}_q^{32}$ and $\mathbb{F}_q^{40}$.

van Dijk-Granger-Page-Rubin-Silverberg-Stam-Woodruff (2005)

We obtain an "almost bijection" between $G_n \times \mathbb{F}_q^{n/3-\varphi(n)}$ and $\mathbb{F}_q^{n/3}$, and thus an "almost bijection" between $G_{30} \times \mathbb{F}_q^2$ and $\mathbb{F}_q^{10}$.

We obtain this using CEILIDH and the equation:

$$\Phi_n(x) \prod_{i=2}^{r-1} \Phi_{p_1 \cdots p_i}(x^{p_{i+2} \cdots p_r}) = \Phi_{p_1 p_2}(x^{p_3 \cdots p_r})$$

for $n = p_1 \cdots p_r$ a product of $r \geq 2$ distinct primes.

Attacks on Discrete Log in $\mathbb{F}_{q^n}^\times$

Granger and Vercauteren (2005), using ideas of Gaudry in the elliptic curve case, exploit the efficient representation of elements of G_2 and G_6 to mount a new kind of attack on the Discrete Logarithm Problem in fields of the form $\mathbb{F}_{q^{2m}}^\times$ and $\mathbb{F}_{q^{6m}}^\times$ for large m.

Torus-based Cryptography

Diffie-Hellman, El Gamal, the $\mathbb{T}_2$ cryptosystem, and CEILIDH are based on certain algebraic groups, i.e., groups that are defined by polynomials. This allows these systems to make full use of the multiplication.

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The Lucas-based cryptosystems and XTR are based on algebraic varieties that are *not* groups. They are quotients of algebraic tori by the action of symmetric groups.

Weil Restriction of Scalars

From now on:

L/k is a cyclic extension of square-free degree n.

E.g., $k = \mathbb{F}_q$, $L = \mathbb{F}_{q^n}$.

G is a commutative algebraic group over k (e.g., the multiplicative group $\mathbb{G}_m$ or an elliptic curve).

$\mathrm{Res}_{L/k}G$ is a commutative algebraic group over k such that

$$(\mathrm{Res}_{L/k}G)(k) = G(L).$$

Example: $\mathrm{Res}_{\mathbb{F}_9/\mathbb{F}_3}\mathbb{G}_m$

$$\mathbb{G}_m(F) = F^\times$$

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The multiplicative group $\mathbb{G}_m$ is defined by $xy = 1$.

Write
$\mathbb{F}_9 = \mathbb{F}_3(\sqrt{-1}), x = x_1 + x_2\sqrt{-1}, y = y_1 + y_2\sqrt{-1}$,
substitute into $xy = 1$, and equate coefficients to obtain defining equations for $\mathrm{Res}_{\mathbb{F}_9/\mathbb{F}_3}\mathbb{G}_m$:

$$x_1y_1 - x_2y_2 = 1, \qquad x_1y_2 + x_2y_1 = 0.$$

Primitive Subgroups

The primitive subgroup $G_{L/k} \subset \mathrm{Res}_{L/k}G$ is

$$G_{L/k} := \ker\Big[\mathrm{Res}_{L/k}G \xrightarrow{\oplus \mathrm{N}_{L/F}} \bigoplus_{k\subseteq F\subsetneq L} \mathrm{Res}_{F/k}G\Big].$$

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$$\mathrm{Res}_{L/k}G \sim \bigoplus_{k\subseteq F\subseteq L} G_{F/k} = G \times \cdots \times G_{L/k}$$

Example: $(\mathbb{G}_m)_{\mathbb{F}_9/\mathbb{F}_3}$

$$\begin{aligned} \mathrm{Res}_{\mathbb{F}_9/\mathbb{F}_3}\mathbb{G}_m &\sim (\mathbb{G}_m)_{\mathbb{F}_3/\mathbb{F}_3} \times (\mathbb{G}_m)_{\mathbb{F}_9/\mathbb{F}_3} \\ &= \mathbb{G}_m \times \ker(\mathrm{N}_{\mathbb{F}_9/\mathbb{F}_3}) \end{aligned}$$

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$$(x_1, x_2, y_1, y_2) \longmapsto (x_1^2 + x_2^2, x_1y_1 + x_2y_2 + 2x_2y_1\sqrt{-1})$$

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$(\mathbb{G}_m)_{\mathbb{F}_9/\mathbb{F}_3}$ is a one-dimensional algebraic torus.

Algebraic tori

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From now on, let $L = \mathbb{F}_{q^n}$ and $k = \mathbb{F}_q$, and let

$$\mathbb{T}_{q,n} = (\mathbb{G}_m)_{L/k}.$$

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$\mathbb{T}_{q,n}(\mathbb{F}_q) = G_n$, the subgroup of $\mathbb{F}_{q^n}^\times$ of order $\Phi_n(q)$.

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The conjectured systems in "Looking beyond XTR" are based on $\mathbb{T}_{q,30}/(S_2 \times S_3 \times S_5)$ and $\mathbb{T}_{q,30}/(S_3 \times S_5)$.

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Fact (Klyachko): The tori $\mathbb{T}_{q,n}$ are rational (i.e., birationally isomorphic to affine space) whenever n is divisible by at most 2 distinct primes.

Voskresenskii's Conjecture

The tori $\mathbb{T}_{q,n}$ are rational; i.e., there is a birational isomorphism

$$\mathbb{T}_{q,n} \quad \dashleftarrow\!\!\dashrightarrow \quad \mathbb{A}^{\varphi(n)}.$$

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Not known for $n = 30$.

Asymptotically Optimal Communication for Torus-Based Cryptography

M. van Dijk and Woodruff got around Voskresenskii's Conjecture by using only that the torus $\mathbb{T}_{q,n}$ is stably rational, i.e., $\mathbb{T}_{q,n} \times \mathbb{A}^A$ is birationally isomorphic to $\mathbb{A}^B$ for some A and B.

Elliptic Curves

An elliptic curve E over a finite field $\mathbb{F}_q$ is a nonsingular curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with the a_i's in $\mathbb{F}_q$.

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with the a_i's in $\mathbb{F}_q$.

One can define a natural addition law on the $\mathbb{F}_q$-solutions so that, along with the point at infinity, they form an abelian group $E(\mathbb{F}_q)$ of order $\approx q$.

Elliptic Curve Cryptography

- V. Miller, 1985.
- N. Koblitz, 1987.

In place of the group $\mathbb{F}_q^\times$, use $E(\mathbb{F}_q)$
(in Diffie-Hellman, El Gamal, etc.).

To "compress" an elliptic curve point (x, y), drop the y-coordinate and send a bit that allows y to be reconstructed from x and the equation for E.

Rubin-Silverberg EC Point Compression

E an elliptic curve over k, $n = [L : k]$ prime:

$$\mathrm{Res}_{L/k} E \sim E \times E_{L/k} = E \times \ker(\mathrm{Tr}_{L/k})$$

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If $P = (x, y) \in E_{L/k}(k) \subset E(L)$, write
$x = \sum_{i=1}^{n} x_i e_i$ wrt a basis $\{e_1, \ldots, e_n\}$ for L/k.

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Compress P to $(x_2, \ldots, x_n)$, and send a few extra bits that allow reconstruction of P from $(x_2, \ldots, x_n)$, the equation for E, and the equation $\mathrm{Tr}(P) = 0$.

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Compress P to $(x_2, \ldots, x_n)$, and send a few extra bits that allow reconstruction of P from $(x_2, \ldots, x_n)$, the equation for E, and the equation $\mathrm{Tr}(P) = 0$.

We can do this efficiently when $n = 2, 3$, or 5.

Rubin-Silverberg EC Point Compression

In other words, we are finding explicit rational maps of small degree

$$E_{L/k} \longrightarrow \mathbb{A}^{n-1}$$

over k.

Cryptographic Security

If G is a commutative algebraic group over $k = \mathbb{F}_q$, let $c(G, k)$ be the smallest c such that every cyclic subgroup of $G(k)$ embeds in $\mathbb{F}_{q^c}^{\times}$.

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$c(G_{L/k}, k) = c(G, L)$ for $G = \mathbb{G}_m$ or a supersingular elliptic curve E over k with $(n, 2qc(E, k)) = 1$, where $n = [L : k]$.

Thus in these cases, the subgroup $G_{L/k}(k)$ is as secure as $G(L)$ against the classical subexponential attacks on the Discrete Logarithm Problem in finite fields, and is smaller.

Abelian Varieties

The fact that this works in the elliptic curve case can be proved using abelian varieties.

An abelian variety is a connected projective algebraic group.

The one-dimensional abelian varieties are exactly the elliptic curves.

Our results were inspired by earlier work of S. Galbraith.

Attacks on Discrete Log in $E(\mathbb{F}_{q^n})$

Gaudry recently exploited the Weil restriction of scalars to mount a new kind of attack on the Discrete Logarithm Problem in $E(\mathbb{F}_{q^n})$ for certain n.

The future

- Find a birational isomorphism between $\mathbb{T}_{q,30}$ and $\mathbb{A}^8$.
- Study attacks on the Discrete Logarithm Problem that use the torus structure (understand current ones and find new ones).
- Find good decompression algorithms for primitive subgroups of elliptic curves when $n > 5$.

Some references

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Applying number theory and algebraic geometry to cryptography

Alice Silverberg

October 8, 2005

AMS Meeting, Bard College