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SIMPLEXWISE LINEAR NEAR-EMBEDDINGS OF A 2-DISK INTO $\mathbf{R}^2$

BY

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ABSTRACT. Let $K \subset \mathbf{R}^2$ be a finitely triangulated 2-disk; a map $f\colon K \to \mathbf{R}^2$ is called *simplexwise linear* (SL) if $f|\sigma$ is affine linear for each (closed) simplex σ of K. Interest in SL maps originated with work of S. S. Cairns and subsequent work of R. Thom and N. H. Kuiper. Let $E(K) = \{$orientation preserving SL embeddings $K \to \mathbf{R}^2\}$, $L(K) = \{$SL homeomorphism $K \to K$ fixing ∂K pointwise$\}$, and $\overline{E(K)}$, $\overline{L(K)}$ denote their respective closures in the space of all SL maps $K \to \mathbf{R}^2$ and the space of all SL maps $K \to K$ fixing ∂K. The main result of this paper is useful characterizations of maps in $\overline{L(K)}$ and some maps in $\overline{E(K)}$, including the relation of such maps to SL embeddings into the nonstandard plane.

1. Definitions and statement of results. Let K be a finite (rectilinear) simplicial complex in $\mathbf{R}^n$; we regard simplices as closed, and will write K when we mean the topological space $|K|$ underlying K. Let K^i denote the set of (closed) i-simplices of K, and when K is a manifold let $(\operatorname{int} K)^0$ and $(\partial K)^0$ denote the interior and boundary vertices of K, respectively. We will study maps of the following type.

DEFINITION. For K as above, a (continuous) map $f\colon K \to \mathbf{R}^m$ is called *simplexwise linear*, abbreviated SL, if the restriction $f|\sigma$ of f to each simplex $\sigma \in K$ is an affine linear map. (Some authors refer to simplexwise linear maps as "linear maps," for example **[BS1** and **Ho1]**.)

From now on let K be a (finitely triangulated) 2-disk in $\mathbf{R}^2$. Of primary interest are the following two spaces of maps.

DEFINITION. $E(K) = \{$orientation preserving SL embeddings $K \to \mathbf{R}^2\}$, $L(K) = \{$SL homeomorphisms $K \to K$ fixing ∂K pointwise$\}$.

REMARKS. An SL map is uniquely determined by its values on vertices. If K has vertices $\{v_1,\ldots,v_p\}$, then the space of all SL maps $K \to \mathbf{R}^2$ is identified with $\mathbf{R}^{2p}$ via the correspondence $f \leftrightarrow (f(v_1),\ldots,f(v_p))$, and $E(K)$ is identified with an open subset of $\mathbf{R}^{2p}$; if K has k interior vertices, then $L(K)$ is identified with an open subset of $\mathbf{R}^{2k}$. We use the norm on $\mathbf{R}^{2p} = \mathbf{R}^2 \times \cdots \times \mathbf{R}^2$ given by

$$\|(X_1,\ldots,X_p)\| = \sup\{\|X_i\| \mid i = 1,\ldots,p\},$$

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so that, for SL maps f, g: $K \to \mathbf{R}^2$,

$$\|f - g\| = \sup\{\|f(v) - g(v)\| \mid v \in K^0\}.$$

Since $E(K)$ and $L(K)$ are identified with subsets of Euclidean spaces, their closures $\overline{E(K)}$ and $\overline{L(K)}$ are well defined.

DEFINITION. A map in $\overline{E(K)}$ is called a *near-embedding*.

Interest in SL maps, and especially $L(K)$, started with the work of S. S. Cairns [**C**] (1944), R. Thom [**T**] (1958) and N. H. Kuiper [**K**] (1965). Various results on $L(K)$ and $E(K)$ were obtained by C.-W. Ho in [**Ho1**] (1973) and [**Ho2**] (1979) and by R. H. Bing and M. Starbird in [**BS1** and **BS2**] (1978). Consult [**BCH** and **CHHS**] for more detailed expostions of these and related results. Recently, R. Connelly, D. W. Henderson and the author [**BCH**] showed that if K is convex, then $L(K)$ is homeomorphic to $\mathbf{R}^{2k}$. Unlike the previous results, it became necessary in the proof of this last result to make use of maps in $\overline{L(K)}$ in a crucial way. It therefore became desirable to find ways of verifying whether a given SL map is in $\overline{L(K)}$ or not, and to understand the topological nature of the boundary of $\overline{L(K)}$. The analog of Theorem 1.2 for $\overline{L(K)}$ answers the first problem, and the analog of Corollary 7.3 for $\overline{L(K)}$, together with the study of near-embeddings used to prove Theorem 1.2, are partial answers to the second. In [**H**] D. W. Henderson will use some ideas in the proof of Theorem 1.2 in his study of simplexwise geodesic homeomorphisms of the 2-sphere. The author, in [**B**], will apply Theorem 1.2 to the study of strictly convex SL embeddings and near-embeddings $K \to \mathbf{R}^2$.

We are also interested in the infinitesimal analog of $E(K)$. Let $^*\mathbf{R}$ denote the nonstandard real numbers (as in [**D**], for example). If σ is a simplex of K, a map $\sigma \to (^*\mathbf{R})^2$ is called *affine linear* if the usual definition using barycentric coordinates holds. Thus, one can discuss SL maps $K \to (^*\mathbf{R})^2$. Let $^\circ$: $^*\mathbf{R} \to \mathbf{R}$, $x \mapsto {}^\circ x$, denote the "standard" part of a number (where $^\circ$ is not defined on infinite numbers); for any SL map f: $K \to (^*\mathbf{R})^2$ with $f(K)$ finite, the map $^\circ f$: $K \to \mathbf{R}^2$ is then an SL map in the usual sense. If $\delta = \langle a, b, c\rangle$ is a positively oriented 2-simplex and f: $K \to (^*\mathbf{R})^2$ is SL, then we write

$$\det(f|\delta) = \det\begin{pmatrix} 1 & f(a) \\ 1 & f(b) \\ 1 & f(c) \end{pmatrix};$$

$\det(f|\delta)$, which is in $^*\mathbf{R}$, can be regarded as twice the signed area of $f(\delta)$, and is independent of the order of the vertices a, b, c as long as the order is compatible with orientation. (Of course, the same definition holds for SL maps $K \to \mathbf{R}^2 \subset (^*\mathbf{R}^2)$.) Determinants provide the simplest way to define the infinitesimal analog of $E(K)$.

DEFINITION. $E(K, (^*\mathbf{R})^2) = \{f\colon K \to (^*\mathbf{R})^2 \mid f \text{ is SL}, f(K) \text{ is finite and } \det(f|\delta) > 0\ \forall\delta \in K^2\}$.

The following space of maps is convenient to work with and is used throughout the paper.

DEFINITION. $R(K) = \{f\colon K \to \mathbf{R}^2 \mid f$ is SL, $f|\partial k$ is an orientation preserving embedding, and for any $q \in f(K)$ there is at most one $\sigma \in K^2$ such that $f(\sigma)$ is a 2-simplex and $f^{-1}(q) \cap \operatorname{int}\sigma \neq \varnothing\}$.

REMARK. A map being on $R(K)$ simply means that besides the condition on ∂K, the images of "noncollapsed" 2-simplices do not intersect in their interiors.

The following lemma (which is proved using a degree argument similarly to [**Ho3**, Theorem 3.2 and **BCH**, Lemma 4.1]), which is useful later on, shows that $E(K, (^*\mathbf{R})^2)$ is really the right generalization of $E(K)$ to the nonstandard case and that $R(K)$ is a reasonable space to work with.

LEMMA 1.1. (i) $E(K) = \{f\colon K \to \mathbf{R}^2 | f$ *is SL*, $f|\partial K$ *is an orientation preserving embedding and* $\det(f|\delta) > 0\ \forall\delta \in K^2\}$, *and*

(ii) $R(K) = \{f\colon K \to \mathbf{R}^2 | f$ *is SL*, $f|\partial K$ *is an orientation preserving embedding and* $\det(f|\delta) \geqslant 0\ \forall\delta \in K^2\}$.

REMARK. From the preceeding lemma it is seen that

$$E(K) \subset \overline{E(K)} \subset R(K).$$

In fact, both inclusions may be proper (depending on K, of course); for the first inclusion this is evident, and for the second this is seen in [**BCH**, Figure 3.2], which shows a map in $R(K)$ not in $\overline{E(K)}$. On the other hand, both $\overline{E(K)}$ and $R(K)$ are closed subsets of $\mathbf{R}^{2p}$, containing $E(K)$ in their interiors and with topological boundaries coinciding in some "nice" parts (see [**BCH**, §4] for more details).

With the above definitions, we now state the main result of this paper, which is a characterization of certain near-embeddings. Let $\varepsilon\colon R(K) \to \mathbf{R}^+$ be defined by

$$\varepsilon(f) = \tfrac{1}{2}\inf\{\|f(v) - f(w)\|\ |v, w \in K^0, f(v) \neq f(w)\}.$$

THEOREM 1.2. *For an SL map* $f\colon K \to \mathbf{R}^2$ *such that* $f|\partial K$ *is an orientation preserving embedding, the following are equivalent*:

(1) $f \in \overline{E(K)}$;

(2) f *is a near-topological embedding* (*i.e.* f *is the limit of topological embeddings*);

(3) $f \in R(K)$ *and* f *is within* $\varepsilon(f)$ *of a topological embedding*;

(4) $f = {}^\circ g$ *for some* $g \in E(K, (^*\mathbf{R})^2)$;

(5) $f \in R(K)$ *and* $f^{-1}f(v)$ *is simply connected for all* $v \in K^0$;

(6) *for each* 1-*simplex* $A \in K^1$ *and any* $x_A \in \operatorname{int} A$ *such that* $f^{-1}f(x_A) \cap K^0 = \varnothing$, $f^{-1}f(x_A)$ *is simply connected, and for each* $\delta \in K^2$ *and any* $x_\delta \in \operatorname{int}\delta$ *such that* $f^{-1}f(x_\delta) \cap K^1 = \varnothing$, $f^{-1}(x_\delta)$ *is connected.*

REMARK. (1) In Theorem 1.2 the hypothesis that $f|\partial K$ is an embedding does not seem to be necessary, but makes the proof much easier and is sufficient for the applications of the theorem in [**B** and **H**]; in [**B**] the very explicit nature of the proof of the theorem is used.

(2) Condition (3) in Theorem 1.2 states that to verify if a map f is in $\overline{E(K)}$, one need not find a sequence of maps in $E(K)$ converging to f, but only a single map in $E(K)$ sufficiently close to f, if f is known to be in $R(K)$ (which is relatively easy to verify).

(3) There are simple examples which show that the condition $f \in R(K)$ in (5) cannot be dropped.

(4) If K is strictly convex, the analog of Theorem 1.2 holds for $L(K)$, $\overline{L(K)}$ and the appropriate analog of $R(K)$ (of course $f|\partial K$ is automatically an orientation

preserving embedding in this case); the proof (which will not be given) is the same as that of Theorem 1.2, except that one must verify that ∂K need not be moved during any step of the proof.

The outline of the paper is as follows: §2 discusses some basic properties of SL maps; §§3–5 discuss various types of subcomplexes of K which are parts of point or line inverses; §6 discusses partial orderings of vertices and 1-simplices of K given by "reasonable" SL maps $K \to \mathbf{R}^2$; §7 contains the main technical proof of the paper and §8 contains the proof of Theorem 1.2.

2. Basic properties of collapsing.

DEFINITION. An SL map $f\colon K \to \mathbf{R}^2$ is called *boundary-nice* if $f|\partial K$ is an orientation preserving embedding and $f(\operatorname{int} K)$ is contained in the interior of the region bounded by $f(\partial K)$.

In §§3–6 we will assume all maps are boundary-nice, thus avoiding special cases involving subcomplexes of K intersecting $f(\partial K)$.

The following definition states all possible generic ways in which a 2-simplex can be mapped affine linearly.

DEFINITION. For an SL map $f\colon K \to \mathbf{R}^2$ and $\delta = \langle a, b, c\rangle \in K^2$, δ is either:

(1) *not collapsed* if $f(\delta)$ is a 2-simplex,

(2) *of type PC* ("point collapse") if $f(\delta)$ is a point,

(3) *of type EC* ("end collapse") if $f(a) = f(b) \neq f(c)$ for some labeling of the vertices of δ (so that $f(\delta)$ is a line segment), or

(4) *of type SC* ("side collapse") if $f(\delta)$ is a line segment but is not of type *EC* (i.e. not two vertices are mapped to the same point).

NOTE. If δ is of type EC or SC, it can be decomposed into level sets (i.e. sets which are mapped to the same point) which are parallel line segments.

DEFINITION. For an SL map $f\colon K \to \mathbf{R}^2$, $A \in K^1$ and $x \in \operatorname{int} A$ with $f^{-1}f(x) \cap K^0 = \varnothing$, we call $f^{-1}f(x)$ an *edge-point-inverse*. (We will write *f-edge-point-inverse* if more than one map is involved.)

The following lemma is immediate.

LEMMA 2.1. *For an SL map $f\colon K \to \mathbf{R}^2$, any edge-point-inverse is the disjoint union of compact* 0- *and* 1-*manifolds* (*possibly with boundary*). □

DEFINITION. An SL map $f\colon K \to \mathbf{R}^2$ is called *ordered* if every edge-point-inverse is simply connected. Let

$$\operatorname{OBR}(K) = \{ f \in R(K) | f \text{ is boundary-nice and ordered}\}.$$

REMARK. If an SL map $f\colon K \to \mathbf{R}^2$ is boundary-nice and ordered, then every component of an edge-point-inverse is an arc (or a point, which we consider to be a degenerate arc), with each endpoint lying in the boundary of a (unique) noncollapsed 2-simplex (although it is not a vertex). If f is also in $R(K)$, then for each such 2-simplex γ, $\det(f|\gamma) > 0$.

LEMMA 2.2. *If $f \in \operatorname{OBR}(K)$, then all edge-point-inverse are connected.*

PROOF. Let λ be a component of an edge-point-inverse $f^{-1}f(x)$ and let α, β be the noncollapsed 2-simplices of K which contain the endpoints of λ. $f(\alpha) \cup f(\beta)$

contains an open neighborhood of $f(\lambda) = f(x)$. If μ were another component of $f^{-1}f(x)$ with its endpoints in (noncollapsed) $\gamma, \delta \in K^2$, it is easy to see that no two of $\alpha, \beta, \gamma, \delta$ are the same. It now follows immediately that having two distinct commponents of $f^{-1}f(x)$ contradicts the definition of $R(K)$, and thus $f^{-1}f(x)$ is connected. □

LEMMA 2.3. *Let $f \in \mathrm{OBR}(K)$. If $M \subset K$ is a subcomplex which is the closure of an open 2-disk and $f(\mathrm{bd}\, M)$ is a point or a line segment, then $f(M) \subset f(\mathrm{bd}\, M)$ (where* "bd" *denotes* mod 2 *boundary).*

PROOF. First, assume the lemma has been proved when M is a 2-disk, and we will deduce the general case. Although arbitrary M need not be a 2-disk, since int M is an open 2-disk, we can find a polygonal circle S very close to bd M, such that S transversally intersects the interiors of all the 1-simplices of M that meet bd M, and the vertices of S are exactly at such intersections. See Figure 2.1. Let N be the closed region bounded by S, which is a polygonal 2-disk. Triangulate $K \cup S$ by adding a single diagonal 1-simplex to each truncated 2-simplex of K; let K' be this triangulation, so that K' subdivides K and N is a subcomplex of K'. Let $\hat{f}: K' \to \mathbf{R}^2$ be the SL map defined as follows: If $v \in (K')^0$ is in K^0, then let $\hat{f}(v) = f(v)$; if not, then $v \in S$ and v corresponds to a unique vertex $u \in (\mathrm{bd}\, M)^0 \subset K^0$ (since each vertex of S is on a 1-simplex of M which meets bd M and is closer to one of the endpoints of this 1-simplex if it spans M), so let $\hat{f}(v) = f(u)$. See Figure 2.1. One can check that $\hat{f} \in \mathrm{OBR}(K')$. K', $\hat{f}$ and N now satisfy the hypotheses of the lemma with N a 2-disk, so $\hat{f}(N) \subset \hat{f}(\partial N)$. However, $f(\mathrm{bd}\, M) = \hat{f}(\partial N)$ and all the vertices of M not in N are in bd M, so that $f(M) \subset f(\mathrm{bd}\, M)$ follows.

Now suppose M is a 2-disk. First, we note that all 2-simplices of M are collapsed by f; since $f(\partial M)$ is a line segment or a point, this fact is trivial for $\beta \in M^2$ if $f(\beta) \subset f(\partial M)$, and for $\beta \in M^2$ with $f(\beta) \not\subset f(\partial M)$ it follows from a straightforward degree argument using the fact that $f \in R(K)$. Now, suppose $f(M) \not\subset f(\partial M)$. Since M is connected and all 2-simplices of M are collapsed by f, $\overline{f(M) - f(\partial m)}$ is the union of finitely many line segments; let L be such a line segment. Since $f(K^0)$

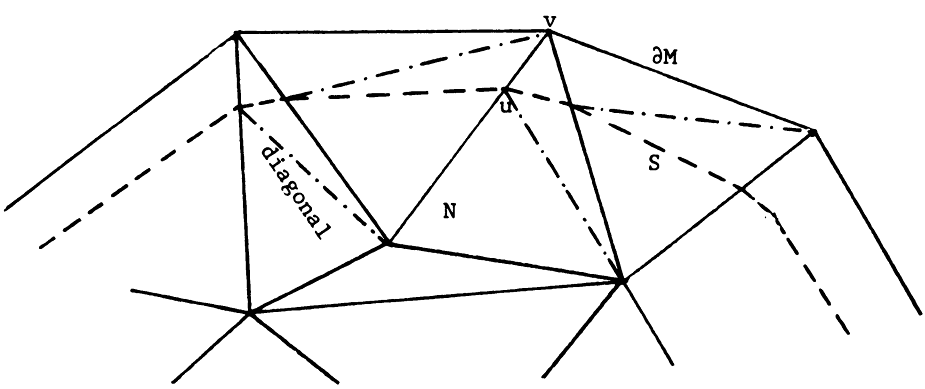

FIGURE 2.1

is a finite number of points, we can pick some $y \in L$ such that $f^{-1}(y) \cap K^0 = \varnothing$. By hypothesis on f and Lemma 2.2, $f^{-1}(y)$ is an arc or a point; consequently, the endpoints of $f^{-1}(y)$ are in the boundaries of noncollapsed 2-simplices of K. However, since $f^{-1}(y) \cap \partial M = \varnothing$ (and hence $f^{-1}(y) \subset \operatorname{int} M$) and since all 2-simplices of M are collapsed, it could not be that $f^{-1}(y)$ intersects a noncollapsed 2-simplex, a contradiction. Hence $f(M) \subset f(\partial M)$. □

3. *f*-segment complexes. Throughout this section, as well as §§4–6, we will let $f \in \mathrm{OBR}(K)$ be fixed.

DEFINITION. $\delta, \gamma \in K^2$ of types EC and/or SC are *f-related* if $f(\gamma) \cap f(\delta)$ contains more than one point (i.e. this intersection is a line segment). We write this relation $\gamma \operatorname{rel} \delta$.

f-relatedness is reflexive and symmetric; let *f-equivalence* be the equivalence relation generated by *f*-relatedness; thus $\gamma, \delta, \varepsilon, K^2$ are *f*-equivalent, written $\gamma \sim \delta$, iff there exists a finite sequence $\varepsilon_1, \ldots, \varepsilon_n \in K^2$ such that

$$\gamma = \varepsilon_1 \operatorname{rel} \varepsilon_2 \operatorname{rel} \cdots \operatorname{rel} \varepsilon_n = \delta.$$

DEFINITION. For $\gamma \in K^2$ of type EC or SC, let

$$\hat{\Lambda}(\gamma) = \{\delta | \delta \in K^2 \text{ is of type EC or SC}, \delta \sim \gamma\}.$$

NOTE. (1) $\hat{\Lambda}(\gamma) = \hat{\Lambda}(\delta)$ iff $\gamma \sim \delta$.

(2) $f(\hat{\Lambda}(\delta))$ is a line segment.

Also, $\hat{\Lambda}(\delta)$ may not be all of $f^{-1}f(\hat{\Lambda}(\delta))$.

LEMMA 3.1. *If $f \in \mathrm{OBR}(K)$, then for any $\delta \in K^2$ of type EC or SC, $\hat{\Lambda}(\delta)$ is a connected subcomplex of K.*

PROOF. $\hat{\Lambda}(\delta)$ is the union of (closed) 2-simplices, and it is a subset of a simplicial complex, so it must be a subcomplex of K.

Suppose $\hat{\Lambda}(\delta)$ is not connected. By the definition of *f*-equivalence, the image of any one component must intersect the image of some other component in a line segment; let C and D be such components. Pick $y \in \operatorname{int}(f(C) \cap f(D))$ such that $f^{-1}(y) \cap K^0 = \varnothing$; $f^{-1}(y)$ must have distinct components in each of C and D, contradicting Lemma 2.2. Thus $\hat{\Lambda}(\delta)$ is connected. □

DEFINITION. For $\delta \in K^2$ of type EC or SC, let $\Lambda(\delta)$ be the minimal 1-connected subcomplex of K containing $\hat{\Lambda}(\delta)$. $\Lambda(\delta)$ will be called the *f-segment complex* of δ.

LEMMA 3.2. *Let $f \in \mathrm{OBR}(K)$ and let $\delta \in K^2$ be of type EC or SC. Then*:

(i) *every 2-simplex in $\Lambda(\delta)$, not in $\hat{\Lambda}(\delta)$, is of type PC*,

(ii) $f(\Lambda(\delta)) = f(\hat{\Lambda}(\delta))$ *is a line segment, and*

(iii) $\Lambda(\delta)$ *is a 2-disk*.

PROOF. (i) Since $\hat{\Lambda}(\delta)$ is a finite, connected subcomplex of K, $\Lambda(\delta)$ is obtained from $\hat{\Lambda}(\delta)$ by "plugging up holes". Each hole is a subcomplex of K which is the closure of an open 2-disk. If H is such a hole, then $f(\operatorname{bd} H) \subset f(\hat{\Lambda}(\delta))$; $f(\hat{\Lambda}(\delta))$ is a line segment and $f(\text{bd } H)$ is compact and connected, so $f(\text{bd } H)$ is a line segment or a point. By Lemma 2.3, $f(H) \subset f(\text{bd } H)$, so that every 2-simplex of H is collapsed. If any 2-simplex β of H were of type EC or SC, it would be in $\hat{\Lambda}(\delta)$ (since $f(\beta) \subset f(\text{bd } H) \subset f(\hat{\Lambda}(\delta))$), a contradiction; thus all 2-simplices of H are of type PC.

(ii) This follows from (i).

(iii) From Lemma 3.1, the minimality condition in the definition of $\Lambda(\delta)$ and (i) of this lemma, it follows that $\Lambda(\delta)$ is a 1-connected subcomplex of K which is the union of 2-simplices. Therefore $\Lambda(\delta)$ is the union of maximal 2-disks, any two of which meet in at most one common boundary vertex. Each maximal 2-disk must contain at least one type EC or SC 2-simplex (which is in $\hat{\Lambda}(\delta)$), by the minimality of $\Lambda(\delta)$. The proof that $\Lambda(\delta)$ is only one such 2-disk is the same as the proof of connectivity in Lemma 3.1. □

DEFINITION. Let f and δ be such that $\Lambda(\delta)$ is a 2-disk. A vertex of $\partial\Lambda(\delta)$ which is mapped by f to the relative interior of $f(\Lambda(\delta))$ is called a *side vertex* of $\Lambda(\delta)$; any other vertex of $\partial\Lambda(\delta)$ is an *end vertex* of $\Lambda(\delta)$. Let e_1, e_2 be the two endpoints of $f(\Lambda(\delta))$. We say $\Lambda(\delta)$ is *simple* if it has the following properties:

$\Lambda(\delta)$ is a 2-disk such that $\partial\Lambda(\delta) = E_1 \cup S_1 \cup E_2 \cup S_2$, where E_i, S_i are (closed) polygonal arcs,

the E_i are possibly single vertices,

the S_i contain at least one 1-simplex each,

$f(E_i) = e_i, f(S_i) = f(\Lambda(\delta))$, and no subarc of S_i has this property,

$E_i \cap S_j$ is a single vertex,

$S_1 \cap S_2$ is empty (if neither E_i is a single vertex) or a single vertex (if exactly one E_i is a single vertex) or two vertices (if both E_i are single vertices), and

E_1, S_1, E_2, S_2 cover $\partial\Lambda(\delta)$ when it is traversed one time around in clockwise order. The E_i are called *ends* of $\Lambda(\delta)$ and the S_i *sides*. See Figure 3.1.

REMARK. E_i may not be all of $f^{-1}(e_i) \cap \Lambda(\delta)$, although no 2-simplex in $f^{-1}(e_i) \cap \Lambda(\delta)$ can intersect E_i in a 1-simplex, by the minimality of $\Lambda(\delta)$.

LEMMA 3.3. *If $f \in \text{OBR}(K)$, then*

(i) *all f-segment complexes are simple*, *and*

(ii) *the edge-point-inverses of an f-segment complex are nontrivial arcs with one endpoint in each of its sides.*

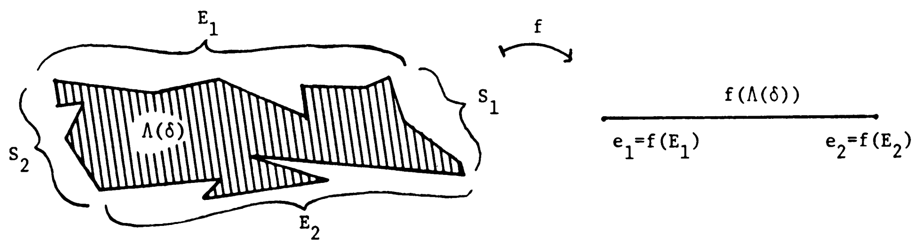

FIGURE 3.1

PROOF. (i) Let $\Lambda(\delta)$ be an f-segment complex; by Lemma 3.2(iii) it is a 2-disk. Lemma 2.4 implies that $f(\Lambda(\delta)) = f(\partial\Lambda(\delta))$. Let $E_1, \ldots, E_n$ be the maximal, connected subcomplexes of $\partial\Lambda(\delta)$ which are mapped to either endpoint of $f(\Lambda(\delta))$; since both endpoints are in $f(\partial\Lambda(\delta))$, $n \geqslant 2$.

The E_i are disjoint, so

$$\partial\Lambda(\delta) = E_1 \cup S_1 \cup E_2 \cup S_2 \cup \cdots \cup E_n \cup S_n \quad \text{for some } n \geqslant 2,$$

where the E_i and S_j have all the properties stated in the definition of simple f-segment complexes with the obvious modifications when $n > 2$. Now, pick any $x \in S_1$ such that $f^{-1}f(x) \cap K^0 = \varnothing$; then $f^{-1}f(x)$ is an edge-point-inverse which intersects all the S_i. By Lemma 2.2, $f^{-1}f(x)$ must be an arc or a point and it follows that $n \leqslant 2$. Hence $n = 2$ and (i) is proved.

(ii) This follows easily from the argument for (i). □

The following lemma shows that the images of simple f-segment complexes under $f \in R(K)$ behave very much like the images of 1-simplices under a homeomorphism.

LEMMA 3.4. *Let $f \in R(K)$ be such that all f-segment complexes are simple*, (*in particular, if $f \in \mathrm{OBR}(K)$*).

(i) *If A, B are either distinct f-segment complexes, or distinct noncollapsed* 1-*simplices not in the same f-segment complex, or an f-segment complex and a noncollapsed* 1-*simplex not contained in it, then* $\operatorname{int} f(A) \cap \operatorname{int} f(B) = \varnothing$.

(ii) *If A is either an f-segment complex or a noncollapsed* 1-*simplex and δ is a noncollapsed* 2-*simplex, then* $f(A) \cap \operatorname{int} f(\delta) = \varnothing$.

REMARK. Part (i) of the above lemma implies, in particular, that the images of simple f-segment complexes cannot intersect transversally.

PROOF OF LEMMA 3.4. The lemma follows immediately from the definition of $R(K)$ (which states that distinct, noncollapsed 2-simplices cannot have the interiors of their images overlap), once the following observation is made: If A is as in the statement of the lemma, there is a neighborhood of $\operatorname{int} f(A)$ entirely contained in the images of noncollapsed 2-simplices which intersect the component of A in

$$\hat{A} \cup \{\delta \mid \delta \in K^2 \text{ is of type PC}, f(\delta) \subset f(A)\},$$

where $\hat{A}$ is either A or, if it exists, the f-segment complex containing A. □

4. f-side complexes.

LEMMA 4.1. *Let $f \in \mathrm{OBR}(K)$ and let $\delta \in K^2$ be of type EC or SC. Then*

$$f^{-1}(\operatorname{int} f(\Lambda(\delta))) = [\Lambda(\delta) - E_1 - E_2] \cup M_1 \cup \cdots \cup M_m,$$

where the E_i are the ends of $\Lambda(\delta)$, and the M_i are 1-*connected subcomplexes of K, each containing at least one side vertex w_i of $\Lambda(\delta)$, and contained in*

$$f^{-1}(f(w_i)) - \operatorname{int} \Lambda(\delta).$$

DEFINITION. The M_i in Lemma 4.1 are called the *f-side complexes* of $\Lambda(\delta)$.

REMARKS. (1) Each f-side complex is only an f-side complex of one f-segment complex, by Lemma 3.4(i).

(2) An f-side complex intersects the f-segment complex it is associated with in a subarc (possibly trivial) of one side of the f-segment complex.

PROOF OF LEMMA 4.1. First, let

$$M = f^{-1}(\operatorname{int} f(\Lambda(\delta))) - \operatorname{int} \Lambda(\delta) - \{\operatorname{int} A \mid A \text{ is a noncollapsed side 1-simplex of } \Lambda(\delta)\}.$$

Using Lemma 3.4, it is routine to show that any 1- or 2-simplex of K, which intersects M in its relative interior, must be mapped to a point. It follows that M is a subcomplex of K and that each component of M is mapped to a point. Let $M_1, \ldots, M_m$ be the components of M. Since each simplex of M is mapped to a point, so is each M_i. No M_i can contain a point of $\Lambda(\delta)$ which is not in a side vertex or in a collapsed side 1-simplex. Suppose some M_j were not simply connected; being a subcomplex, it would then have a hole H which is a subcomplex of K, is the closure of an open 2-disk, and is such that $H \cap M_j = \operatorname{bd} H$. $f(\operatorname{bd} H) \subset f(M_j)$, so $f(\operatorname{bd} H)$ is a point. By Lemma 2.3, $f(H) \subset f(\operatorname{bd} H)$, so $f(H)$ is a point, and thus $H \subset M_j$, a contradiction. Hence M_j is 1-connected (being connected by definition).

We now wish to show that each M_i intersects $\Lambda(\delta)$, so suppose otherwise for some M_j. M_j is a finite, full, contractible subcomplex of K, and hence the simplicial neighborhood N of M_j in K (that is, the union of all (closed) 2-simplices of K which intersect M_j) is a subcomplex of K whose interior is an open 2-disk. Also, no vertex of bd N can be mapped to $f(M_j)$, by the maximality of M_j, since each vertex of bd N is the endpoint of a 1-simplex which intersects M_j. On the other hand, it is easy to verify that no 2-simplex of K can intersect both M_j and $\Lambda(\delta)$, and yet be in neither, so that $f(\operatorname{bd} N) \cap f(\Lambda(\delta)) = \varnothing$. Thus, $f(M_j) \subset f(\Lambda(\delta))$ implies $f(M_j) \not\subset f(\operatorname{bd} N)$, so that $f(N) \not\subset f(\operatorname{bd} N)$.

Now, since $f \in R(K)$, it follows that all the 2-simplices intersecting M_j, but not in M_j, are of types EC and SC, or otherwise the interiors of the images of the uncollapsed ones would intersect the interiors of the images of some uncollapsed 2-simplices intersecting $f^{-1}f(\Lambda(\delta))$. $f(N)$ is therefore a line segment. Since bd N is connected, $f(\operatorname{bd} N)$ must be a line segment or a point. Lemma 2.3 now implies that $f(N) \subset f(\operatorname{bd} N)$, contradicting the conclusion of the preceding paragraph. Thus M_j must intersect $\Lambda(\delta)$.

As mentioned previously, $M_i \cap \Lambda(\delta)$ must be a collapsed, connected subcomplex of one side of $\Lambda(\delta)$, or a single side vertex, so in particular $M_i \cap \Lambda(\delta)$ must contain a side vertex w_i. $f(M_i)$ is a point, so $M_i \subset f^{-1}f(w_i)$, and by definition $M_i \cap \operatorname{int} \Lambda(\delta) = \varnothing$, so $M_i \subset f^{-1}f(w_i) - \operatorname{int} \Lambda(\delta)$. Finally,

$$M_1 \cup \cdots \cup M_m = M = f^{-1}(\operatorname{int} f(\Lambda(\delta))) - \operatorname{int} \Lambda(\delta) - \{\operatorname{int} A \mid A \text{ is a noncollapsed side 1-simplex of } \Lambda(\delta)\},$$

so

$$\begin{aligned}[\Lambda(\delta) - E_1 - E_2] \cup M_1 \cup \cdots \cup M_m &= [f^{-1}(\operatorname{int} f(\Lambda(\delta))) - \operatorname{int} \Lambda(\delta) \\ &\quad - \{\operatorname{int} A \mid A \text{ is a noncollapsed side 1-simplex of } \Lambda(\delta)\}] \\ &\quad \cup [\Lambda(\delta) - E_1 - E_2] \\ &= f^{-1}(\operatorname{int} f(\Lambda(\delta))). \quad \square\end{aligned}$$

5. *f*-vertex-inverses.

LEMMA 5.1. *Let $f \in \mathrm{OBR}(K)$ and let $v \in K^0$. Then*

(i) *if $f(v) \in \operatorname{int} f(\Lambda(\delta))$ for a (unique) f-segment complex $\Lambda(\delta)$, then $f^{-1}f(v) \cap \Lambda(\delta)$ is 1-connected, and*

(ii) *if $f(v) \notin \operatorname{int} F(\Lambda(\delta))$ for any f-segment complex $\Lambda(\delta)$, then*

$$f^{-1}(v) - \bigcup\{\operatorname{int} \Lambda(\gamma) | \gamma \in K^2 \textit{ is of type EC or SC}\}$$

is a 1-connected subcomplex of K.

REMARK. In case (i), $f^{-1}f(v) \cap \Lambda(\delta)$ need not be a subcomplex of $\Lambda(\delta)$, although it is the union of a subcomplex and line segments than span type EC or SC 2-simplices.

DEFINITION. If v is as in case (i) or (ii) of Lemma 5.1 (it must be as in one of the cases), then $f^{-1}f(v) \cap \Lambda(\delta)$ or

$$f^{-1}f(v) - \bigcup\{\operatorname{int} \Lambda(\gamma) | \gamma \in K^2 \text{ is of type EC or SC}\},$$

respectively, is the *f-vertex-inverse* of v, denoted $\Gamma(v)$.

PROOF OF LEMMA 5.1. (i) Suppose $\Gamma(v)$ is not simply connected; then there is a polygonal circle C in $\Gamma(v)$ such that the interior of the 2-disk D bounded by C is not entirely contained in $\Gamma(v)$. It follows that there must be a 1-simplex A of $\Lambda(\delta)$ such that $A \cap \operatorname{int} D - \Gamma(v)$ contains a line segment. Pick some $x \in A \cap \operatorname{int} D - \Gamma(v)$ such that $f^{-1}f(x) \cap K^0 = \varnothing$. By Lemma 3.3(ii), $f^{-1}f(x)$ must be an arc with each

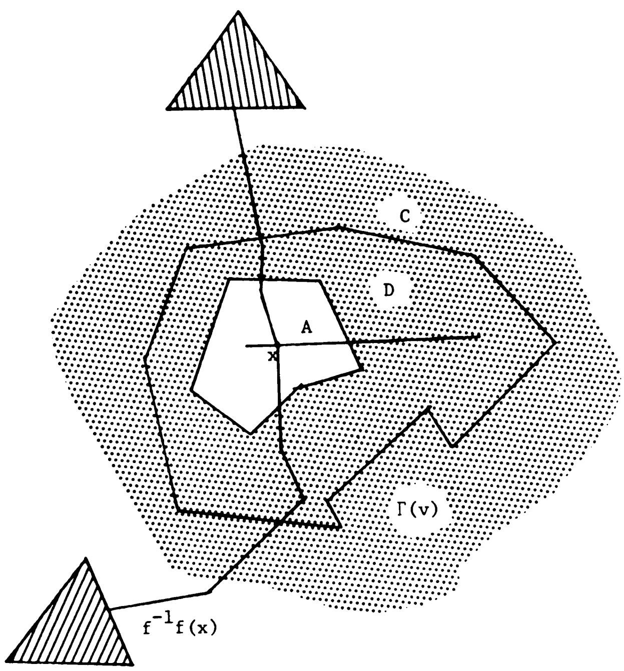

FIGURE 5.1

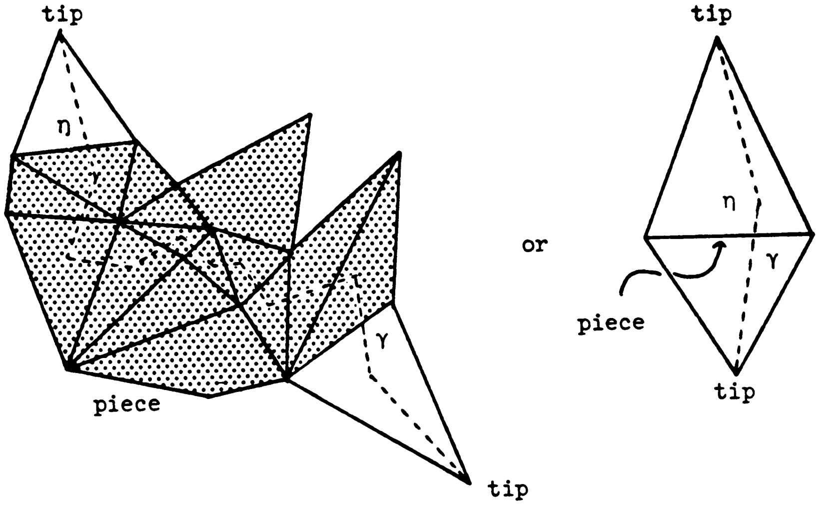

FIGURE 5.2

of its endpoints in the boundary of a noncollapsed 2-simplex, so that $f^{-1}f(x) \cap C \neq \varnothing$ since $\operatorname{int} D \subset \operatorname{int} \Lambda(\delta)$; it follows that $v \in f^{-1}f(x)$, contradicting the choice of x, and hence $\Gamma(v)$ must be simply connected. See Figure 5.1.

To see that $\Gamma(v)$ is connected, first note that $f(v)$ separates $f(\Lambda(\delta))$, so let Δ be one of the components of $f(\Lambda(\delta)) - f(v)$ and let

$$T = \left\{\gamma \in \Lambda(\delta)^2 | \gamma \cap \Gamma(v) \neq 0 \text{ and } f(\gamma) \cap \Delta \text{ is a line segment}\right\}.$$

T is not empty. One can choose $y \in T$ such that $f^{-1}f(y) \cap K^0 = \varnothing$ and there is no $w \in K^0$ with $f(w) \in f(\Lambda(\delta))$ between $f(v)$ and $f(y)$. $f^{-1}f(y)$ is an arc by Lemma 3.3(ii) and it intersects every 2-simplex in T. Hence T is mod 2 connected along noncollapsed edges. Since each 2-simplex in T intersects $\Gamma(v)$, it follows easily that $\Gamma(v) \cap T$ is connected. However, every component of $\Gamma(v)$ must intersect T (using the simple connectivity of $\Gamma(v)$ and the hypothesis on f), and it follows that $\Gamma(v)$ is connected.

(ii) First, note that $f^{-1}f(v)$ is a 1-connected subcomplex of K by an argument like that in Lemma 4.1. Now,

$$W = K - \bigcup\{\operatorname{int} \Lambda(\gamma) | \gamma \in K^2 \text{ is of type EC or SC}\}$$

is a subcomplex, so $\Gamma(v) = f^{-1}f(v) \cap W$ is a subcomplex. $\Gamma(v)$ is seen to be connected by an argument like the proof of Lemma 3.4, and simply connected similarly to the simple connectivity of f-side complexes (Lemma 4.1). □

Let $f \in \mathrm{OBR}(K)$ and let $\Gamma(v)$ be an f-vertex-inverse. Being simply connected, $\Gamma(v)$ is the union of maximal 2-disks (which are subcomplexes of K), 1-simplices, and line segments that span type EC or SC 2-simplices; in §7 we will use some of the above types of unionands.

DEFINITION. A unionand of $\Gamma(v)$ as above is called a *piece* of $\Gamma(v)$ if it is either (1) a 2-disk, (2)a 1-simplex (not in $\partial\Lambda(\delta)$ if we are in case (i) of the definition of $\Gamma(v)$), or (3) a spanning line segment that contains a vertex (so that it spans a type SC 2-simplex). If P is a piece of $\Gamma(v)$, then a *tooth* of P is a 2-simplex η (contained in $\Lambda(\delta)$ if we are in case (i) of the definition of $\Gamma(v)$) which intersects P in exactly one 1-simplex if P is of type (1) or (2) above, or is spanned by P if P is of type (3) above. (In the first case η is of type EC, and in the second case type SC.) The vertices of η not in P are called *tips* of P.

REMARKS. (1) Not every unionand of $\Gamma(v)$ is a piece of $\Gamma(v)$, although every f-vertex-inverse contains at least one piece.

(2) Each tooth corresponds to a unique piece.

(3) Every piece has at least two tips.

DEFINITION. Given a piece P of $\Gamma(v)$ and two (distinct) tips u, w of P (contained in teeth η, γ, respectively, which may not be distinct), a *pulling path* for P, u and w is a finite polygonal path $l\colon [0,1] \to \operatorname{int} P \cup \eta \cup \gamma$ such that:

(i) l is injective,

(ii) $l(0) = u$, $l(1) = w$ and $l((0,1)) \cap \partial\eta = \varnothing = l((0,1)) \cap \partial\gamma$,

(iii) $l((0,1))$ contains no vertices, and

(iv) $l([0,1])$ intersects 1-simplices transversally (and P also if P is a spanning line segment), at most once each. See Figure 5.2.

REMARK. Given two distinct tips of a piece there is always a (not necessarily unique) pulling path connecting them.

6. Ordering vertices and 1-simplices. Let $f \in \mathrm{OBR}(K)$ and let $\Lambda(\delta)$ be an f-segment complex. Lemma 3.3(i) says that $\Lambda(\delta)$ has two sides; choose one to be called the *top side* and the other the *bottom side*. Suppose, without loss of generality, that the top side is labelled S_1 in the decomposition $\partial\Lambda(\delta) = E_1 \cup S_1 \cup E_2 \cup S_2$ going clockwise around $\partial\Lambda(\delta)$, as in the definition of simple f-segment complexes. Of the two directions perpendicular to $f(\Lambda(\delta))$, let the *positive direction* be the one which, if it coincided with the positive y-axis direction, would make $f(E_2) - f(E_1)$ be in the positive x-axis direction. Let the *positive half-plane* be the component of $\mathbf{R}^2 - \{\text{line containing } f(\Lambda(\delta))\}$ corresponding to the positive direction. Finally, Lemma 3.3(ii) says that every edge-point-inverse in $\Lambda(\delta)$ has exactly one endpoint in each side of $\Lambda(\delta)$, and we call these endpoints the *top* and *bottom* ones corresponding to which sides they are in.

REMARK. If γ is a noncollapsed 2-simplex of K which intersects the interior of the top side of $\Lambda(\delta)$, then $\operatorname{int} f(\gamma)$ is in the positive half-plane.

Let A, B be distinct, noncollapsed 1-simplices of $\Lambda(\delta)$ such that $f(A) \cap f(B)$ is a line segment. Then for any $x \in f(A) \cap f(B)$ such that $f^{-1}(x)$ is an edge-point-inverse, $f^{-1}(x) \cap A$ and $f^{-1}(x) \cap B$ are distinct points in the arc $f^{-1}(x)$.

DEFINITION. For A, B and x as above, we say A is *above* B if $f^{-1}(x) \cap A$ is closer to the top endpoint of $f^{-1}(x)$ than $f^{-1}(x) \cap B$; we also say B is *below* A. That this definition does not depend on the choice of x is just the initial step in the proof of Lemma 6.1.

NOTE. For $A, B \in \Lambda(\delta)^1$ such that $f(A) \cap f(B)$ is not a line segment, neither A nor B is above the other.

DEFINITION. Let $A \in \Lambda(\delta)^1$ be noncollapsed and let $v \in \Lambda(\delta)^0$ be such that $f(v) \in \operatorname{int} f(A)$; we say v is *above* A if, for any noncollapsed $B \in \Lambda(\delta)^1$ which intersects the component of v in $\Gamma(v) - A$, B is above A. It can be checked, as in the previous definition, that the choice of B does not matter. If v is above A, and A is above all other 1-simplices of $\Lambda(\delta)$ which v is above, then we say v is *immediately above* A.

NOTE. (1) If $A = \langle w, u \rangle$ is above B and $f(w) \in \operatorname{int} f(B)$, then w is above B.

(2) For $v \in \Lambda(\delta)^0$, v need not be above any 1-simplex of $\Lambda(\delta)$; if it is above some 1-simplices, then there is a unique 1-simplex which it is immediately above (by Lemma 6.1).

LEMMA 6.1. *Let $f \in \mathrm{OBR}(K)$ and let $\Lambda(\delta)$ be an f-segment complex with chosen top side. If $A_0, \ldots, A_n$ are distinct, noncollapsed 1-simplices of $\Lambda(\delta)$ such that A_i is above A_{i+1} for $0 \leqslant i \leqslant n-1$, then A_n is not above A_0. In particular, there cannot exist two distinct 1-simplices each above the other.*

PROOF. The proof is by induction on n, where we assume $n \geqslant 1$. For $n = 1$, suppose the lemma is false, i.e. A_1 is above A_0. Since A_0 is above A_1 by hypothesis, there is an edge-point-inverse λ which intersects the interiors of A_0 and A_1 so that $\lambda \cap A_0$ is closer to the top endpoint of λ. Since A_1 is above A_0, there is also an edge-point-inverse μ intersecting the interiors of A_0 and A_1, so that $\mu \cap A_1$ is closer to the top endpoint of μ. Note that μ and λ cannot intersect and that neither can intersect A_0 or A_1 more than once (or once nontransversally). Hence μ cannot intersect A_0 before it intersects A_1 (coming from the top). Using λ, A_1 has a "top" side and a "bottom" side, and μ intersects A_1 either from top to bottom or vice-versa. The bottom-to-top case is pictured in Figure 6.1. Label points a, b, c and d as in the figure. In the bottom-to-top case, it is seen that for μ to intersect the bottom side of $\Lambda(\delta)$ (which it must do), μ must intersect (in a point below c) the circle which is the union of λ from a to b, A_1 from b to c, μ from c to d, and the top side of $\Lambda(\delta)$ from d to a. However, such an intersection cannot happen, so the

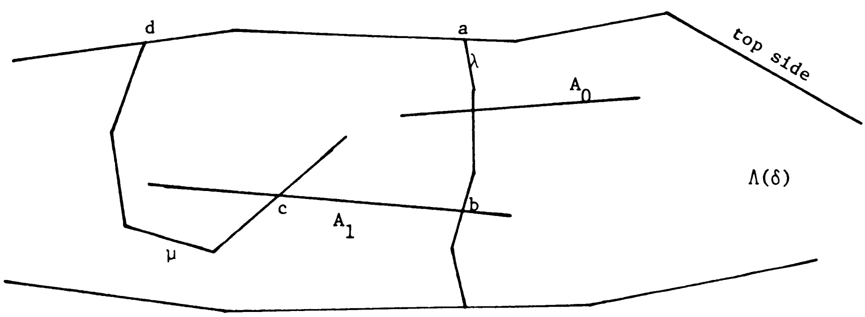

FIGURE 6.1

bottom-to-top case is impossible. A similar contradiction is obtained in the top-to-bottom case, and the lemma holds for $n = 1$.

Now suppose that $n \geqslant 2$ and the inductive hypothesis holds for all cases with fewer than $n + 1$ 1-simplices. For each $0 \leqslant i \leqslant n - 1$, let λ_i be an edge-point-inverse which intersects the interiors of A_i and A_{i+1} so that $\lambda_i \cap A_i$ is closer to the top endpoint of λ_i. Define λ to be the spanning arc of $\Lambda(\delta)$ which is the union of λ_0 from its top to $\lambda_0 \cap A_1$, A_1 between $\lambda_0 \cap A_1$ and $\lambda_1 \cap A_1$, λ_1 from $\lambda_1 \cap A_1$ to $\lambda_1 \cap A_2$, A_2 between $\lambda_1 \cap A_2$ and $\lambda_2 \cap A_2, \ldots, A_{n-1}$ between $\lambda_{n-2} \cap A_{n-1}$ and $\lambda_{n-1} \cap A_{n-1}$, and λ_{n-1} from $\lambda_{n-1} \cap A_{n-1}$ to its bottom. See Figure 6.2. Suppose the lemma is false, so that there exists an edge-point-inverse μ which intersects the interiors of A_0 and A_n so that $\mu \cap A_n$ is closer to the top endpoint of μ. If μ intersected some A_i, $1 \leqslant i \leqslant n - 1$, before it intersected A_n, then A_i would be above A_0, contradicting the inductive hypothesis for $A_0, \ldots, A_i$; if μ intersected such A_i after intersecting A_n, then A_n would be above A_i, contradicting the inductive hypothesis for $A_i, \ldots, A_n$. Hence $\mu \cap A_i = \varnothing$ for $1 \leqslant i \leqslant n - 1$. As before, $\mu \cap \lambda_i = \varnothing$ for all i, and hence $\mu \cap \lambda = \varnothing$. The same analysis as for $n = 1$, when applied to A_0, A_n, λ and μ, shows μ cannot exist, and the lemma is proved. □

The following lemma is straightforward.

LEMMA 6.2. *Let $\gamma = \langle a, b, c\rangle$ be a type EC or SC 2-simplex in $\Lambda(\delta)$, with $\langle a, c\rangle$ above $\langle a, b\rangle$. If $g\colon \gamma \to \mathbf{R}^2$ is an affine linear map with $g(a) = f(a)$, $g(b) = f(b)$ and $g(c)$ in the positive half-plane, then* $\det(g|\gamma) > 0$. *If $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ is any orientation preserving affine linear map, then* $\det(T \circ g|\gamma) > 0$. □

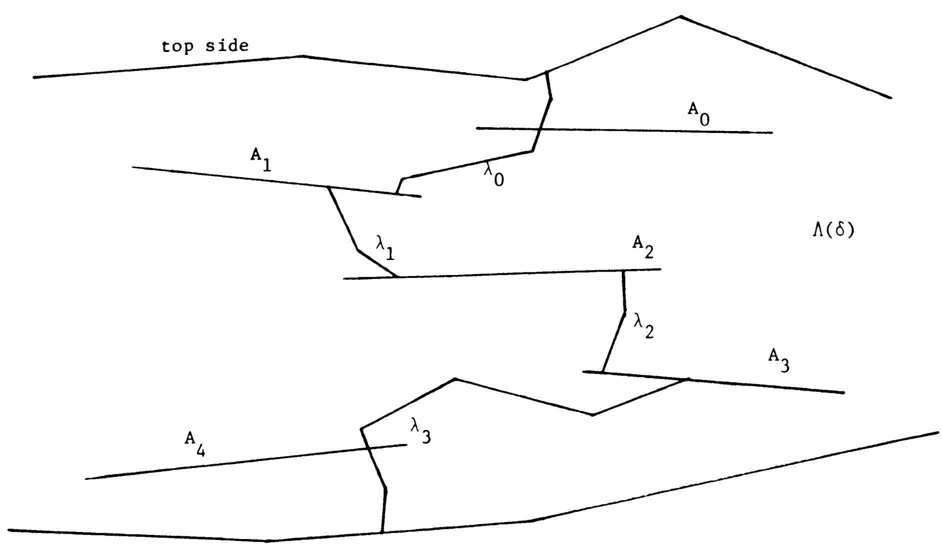

FIGURE 6.2

7. Pulling apart collapses. The main technical result of this paper, from which Theorem 1.2 will be deduced, is the following proposition.

PROPOSITION 7.1. *Let $f \in \mathrm{OBR}(K)$ and $\rho > 0$ be given. Then there is a finite polygonal path f_t: $[0,1] \to \mathrm{OBR}(K)$ of length less than ρ such that $f_0 = f$ and $f_1 \in E(K)$.*

COROLLARY 7.2. *If $f \in R(K)$ is oriented, then $f \in \overline{E(K)}$.*

PROOF. If f is boundary-nice, then the corollary follows immediately from Proposition 7.1. If f is not boundary-nice, then by adding an appropriately triangulated collar to the outside of K and suitably extending f to the collar by an embedding, we can reduce this case to the boundary-nice case. □

COROLLARY 7.3. *If $f \in \overline{E(K)}$ is injective on ∂K, then there is a finite polygonal path in $\overline{E(K)}$ of arbitrarily short length from f to a point in $E(K)$.*

PROOF. This follows immediately from Proposition 7.1, the implication (1) ⇒ (6) in Theorem 1.2 (which does not require Proposition 7.1), and the collaring argument in the proof of Corollary 7.2. □

PROOF OF PROPOSITION 7.1. Let $S(f)$ be the number of 2-simplices collapsed by f; the proof is by induction of $S(f)$. If $S(f) = 0$, it follows from Lemma 1.1 that $f \in E(K)$, and there is nothing to prove. Now assume $S(f) > 0$. We will proceed by constructing a homotopy f_t: $[0,1] \to \mathrm{OBR}(K)$ such that $f_0 = f$, $S(f_1) < S(f)$, and the homotopy is a straight line of length less than $1/2\rho$. Induction will then complete the proof. To define the homotopy we will find a collection of vertices, denoted $V(f)$, such that $f(V(f))$ is a point, and then move the image of $V(f)$ slightly, keeping it a point; other vertices may have their images moved as well, in order to insure that all maps are oriented and in $R(K)$. The length of the homotopy may have to be much less than $1/2\rho$.

Since f is boundary-nice, it is easy to see that not all collapsed 2-simplices are of type PC, and hence there is at least one (nonempty) f-segment complex. By Lemma 3.3(i) all f-segment complexes are simple. We consider two cases.

Case 1. *Some f-segment complex has a side vertex.* Let $\Lambda(\delta)$ be an f-segment complex with side vertex v. Then $\Gamma(v) = f^{-1}f(v) \cap \Lambda(\delta)$ is 1-connected by Lemma 5.1(i). Note that $\Gamma(v) \cap \operatorname{int} \Lambda(\delta) \neq \varnothing$; it follows that $\Gamma(v)$ must have some piece P which contains v (and is not, by definition, a single side 1-simplex in $\partial\Lambda(\delta)$). Moreover, we can pick tips z, w of P such that $f(z)$ and $f(w)$ lie in distinct components of $f(\Lambda(\delta)) - f(v)$. Let l be a pulling path for P, z, w. $l([0,1])$ separates $\Gamma(v)$ into two connected subsets, and let $V(f)$ be the vertices of the subset containing v (and hence $f^{-1}f(v) \cap \partial\Lambda(\delta)$). We will define f_t by specifying its action on vertices; in particular, f_t will move the image of $V(f)$ by starting at $f(V(f))$ and then making a straight line (shorter than $1/2\rho$) into the positive half-plane, at any chosen angle with $f(\Lambda(\delta))$. (Here the choice of angle is irrelevant, but in an application of this proof in **[B]** it will be necessary to note that any particular angle will work.) See Figure 7.1.

We need to determine which other vertices of K need to have their images moved, and how to move them, so that f_t will be oriented and in $R(K)$ (boundary-nice is no problem if f_t moves all vertices by small enough amounts). Only vertices in $f^{-1}f(\text{int }\Lambda(\delta))$ will be moved. We will first discuss the vertices of $\Lambda(\delta)$; we will use the ideas of the previous section to give an ordering to these vertices, thus allowing an inductive definition of $f_t|\Lambda(\delta)^0$. Let the side of $\Lambda(\delta)$ containing v be chosen as the top side.

DEFINITION. A noncollapsed 1-simplex A of $\Lambda(\delta)$ is called *movable* if there is a chain $A = A_0, A_1, \ldots, A_n$ of noncollapsed 1-simplices of $\Lambda(\delta)$ such that A_i is above A_{1+i} for $0 \leqslant i \leqslant n-1$, and A_n intersects $V(f)$ (necessarily in a single endpoint). A vertex $v \in \Lambda(\delta)$ is *movable* if it is above some movable 1-simplex (and hence is immediately above a unique one).

It is easy to see that if A is movable, then $f(V(f)) \notin \text{int } f(A)$, using Lemma 6.1; hence the set of all movable 1-simplices is $\mathcal{M}_R \cup \mathcal{M}_L$, where

$$\mathcal{M}_R = \{A \in \Lambda(\delta)^1 | A \text{ is movable and int } f(A) \text{ is in the right-hand component of } f(\Lambda(\delta)) - f(V(f))\},$$

and similarly for $\mathcal{M}_L$ using left instead of right. Correspondingly, the set of all movable vertices is $\mathcal{V}_R \cup \mathcal{V}_L$, where $v \in \mathcal{V}_R$ iff it is above something in $\mathcal{M}_R$, and similarly for $\mathcal{V}_L$. It follows from Lemma 6.1 that we can order the members of $\mathcal{M}_R$, writing them $A_1, \ldots, A_r$ in order, so that A_i is not above A_j for all $j > i$. The members of $\mathcal{M}_L$ can be similarly ordered, writing them $B_1, \ldots, B_l$ in order. Note that nothing in $\Lambda(\delta)^1 - \mathcal{M}_R$ is above anything in $\mathcal{M}_R$, and similarly for $\mathcal{M}_L$.

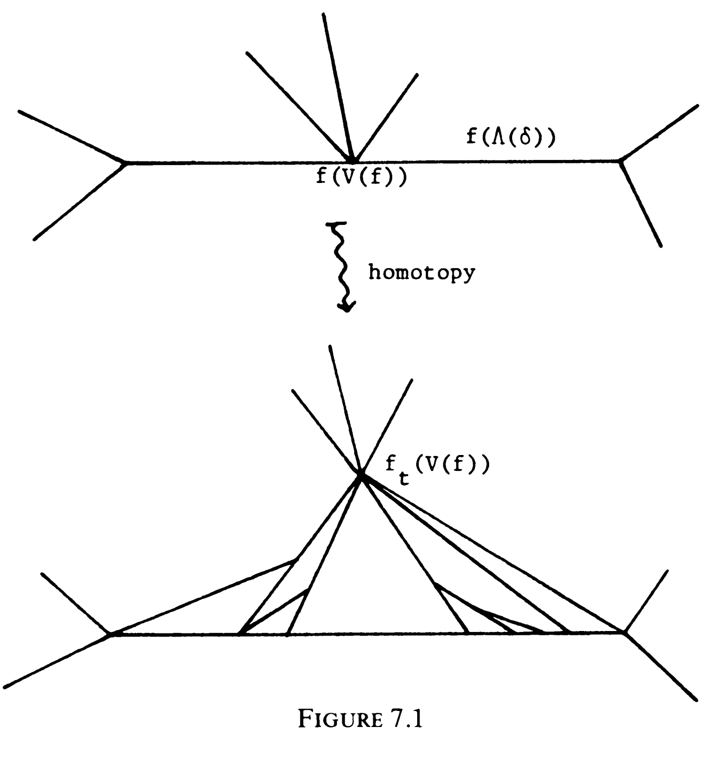

FIGURE 7.1

DEFINITION. For $v \in \Lambda(\delta)^0$, define $N(v)$ by

$$N(v) = \begin{cases} 0 & \text{if } v \in V(f); \\ i & \text{if } v \in \mathscr{V}_R \text{ and } v \text{ is immediately above } A_i, \text{ or} \\ & v \in \mathscr{V}_L \text{ and } v \text{ is immediately above } B_i; \\ -1 & \text{otherwise.} \end{cases}$$

REMARK. If $v \in \Lambda(\delta)$ has $N(v) > 0$ and v is immediately above $A = \langle a, b \rangle$, then $N(a), N(b) < N(v)$.

We now define $f_t(v)$ for $v \in \Lambda(\delta)^0$, inductively on $N(v)$. If $N(v) = -1$, let $f_t(v) = f_0(v)$ for all $t \in [0,1]$. If $N(v) = 0$, then $v \in V(f)$, and $f_t(v)$ has been specified already. Now suppose that $N(v) > 0$ and $f_t(w)$ has been defined for all $w \in \Lambda(\delta)^0$ such that $N(w) < N(v)$. By the definition of $N(v)$, v is immediately above $A_{N(v)} = \langle a, b \rangle$. $N(a), N(b) < N(v)$ by the previous remark, so f_t is defined on $A_{N(v)}$.

We define $f_t(v)$ to be the intersection of the line segment $f_t(A_{N(v)})$ (which is not collapsed if we move $V(f)$ by a very small amount) and the line l which contains $f(v)$ and is parallel to $\langle f_t(v(f)), f_0(V(f)) \rangle$. Note that since $V(f)$ is moved in a straight line by f_t, l does not depend on t, and also that the required intersection exists if $V(f)$ is moved by a very small amount.

For any vertex v in $f^{-1}(\text{int } f(\Lambda(\delta))) - \Lambda(\delta)$, v is in an f-side complex which contains at least one side vertex w of $\Lambda(\delta)$; all such side vertices are mapped by f to the same point and f_t moves them in the same way, so we let $f_t(v) = f_t(w)$. This completes the definition of f_t on all vertices of $f^{-1}(\text{int } f(\Lambda(\delta)))$; f_t fixes all other vertices, and we have defined a continuous map f_t: $[0,1] \to \{\text{SL maps } K \to \mathbf{R}^2\}$.

It is evident from the definition of f_t that the teeth used to define the pulling path for $\Gamma(v)$ are not collapsed by f_t for $t \in (0,1]$, but since they are collapsed by $f_0 = f$, $S(f_t) < S(f)$ for $t \in (0,1]$. (Clearly nothing new is collapsed during the homotopy if f_t moves vertices by small enough amounts.) Also, since f is boundary-nice, it is clear that if f_t is a small enough homotopy, then it is boundary-nice for all t. Hence, to finish the proof of Case 1, it remains to be seen that f_t is oriented and in $R(K)$ for all t.

To show $f_t \in R(K)$, it suffices, by Lemma 3.1, to show that $\det(f_t|\gamma) \geqslant 0$ for all $\gamma \in K^2$. It is evident that for $\gamma \notin \Lambda(\delta)^2$, $\det(f_t|\gamma) = \det(f|\gamma) \geqslant 0$ for a small enough homotopy; hence we need only examine $\gamma \in \Lambda(\delta)^2$. There are a number of cases. If γ is one of the teeth used to define the pulling path, then it can be checked that $\det(f_t|\gamma) > 0$ for $t \in (0,1]$, using the way in which the images of vertices in $V(f)$ are moved and Lemma 6.2. If γ is a type PC 2-simplex (with respect to f) contained in $\Gamma(v)$, then the definition of f_t implies that γ is either of type PC or EC with respect to f_t ($t \in (0,1]$), so that $\det(f_t|\gamma) = 0$. Now, if $B \in \Lambda(\delta)^1$ is mapped by f to a point other than $f(V(f))$, it is seen that $f_t(B)$ is a point for all t. Therefore, if γ is either a type PC 2-simplex not in $\Gamma(v)$, or a type EC 2-simplex which is not one of the teeth of $\Gamma(v)$ used above (both with respect to f), then γ remains of the same type with respect to f_t, and hence $\det(f_t|\gamma) = 0$. Finally, suppose γ is of type SC (with respect to f), so that $\gamma = \langle a, b, c \rangle$ with $f(a) \in \text{int } f(\langle b, c \rangle)$. If a is above

$\langle b, c\rangle$ then it is clearly immediately above, and thus $f_t(a) \in \text{int}\, f_t(\langle b, c\rangle)$ for all t (by the definition of f_t), so $\det(f_t|\gamma) = 0$. The only remaining case is when a is not above $\langle b, c\rangle$; the desired result in this case will follow from the following Claim and Lemma 6.2. First some definitions:

Fix $t \in (0, 1]$. Let us assume that $f(\Lambda(\delta))$ is in the x-axis, $f(V(f))$ is the origin, and the positive half-plane is the standard upper half-plane. The line segment $\langle f_t(V)(f), f_0(V(f))\rangle$ may make any angle in $(0, \pi)$ with $f(\Lambda(\delta))$, but we will assume for convenience that the angle is $\pi/2$, since the obvious modifications of our arguments will work for any angle. Let π_x: $\mathbf{R}^2 \to \mathbf{R}$ be projection onto the x-axis. Note that for movable $A, B \in \Lambda(\delta)^1$, if the line segments $f_t(A)$ and $f_t(B)$ do not intersect in their interiors and if $\pi_x(f_t(A)) \cap \pi_x(f_t(B))$ is a line segment, then either for all $p \in \text{int}[\pi_x(f_t(A)) \cap \pi_x(f_t(B))]$, $\pi_x^{-1}(p) \cap f_t(A)$ has larger y-coordinate than $\pi_x^{-1}(p) \cap f_t(B)$, or, for all such p, the opposite inequality of y-coordinates holds; in the first case we say $f_t(A)$ is *Euclideanly-above* $f_t(B)$, and vice-versa in the second. See Figure 7.2.

In this paragraph and in the following claim, we will discuss some properties of the images (under f_t) of the movable 1-simplices. All such 1-simplices are either in $\mathscr{M}_R$ or $\mathscr{M}_L$ (but not both), so we will only discuss $\mathscr{M}_R$, since $\mathscr{M}_L$ is exactly the same. Let e_R be the right endpoint of $f_t(\Lambda(\delta))$, let T_R be the triangle with vertices $\{e_R, f(V(f)), f_t(V(f))\}$, and let

$$D_i = T_R - \bigcup_{k=1}^{i} \{\text{trapezoid between } f_t(A_k) \text{ and } f(\Lambda(\delta))\}$$

for $1 \leqslant i \leqslant r$, where $\mathscr{M}_R = \{A_1, \ldots, A_r\}$ as before. See Figure 7.3.

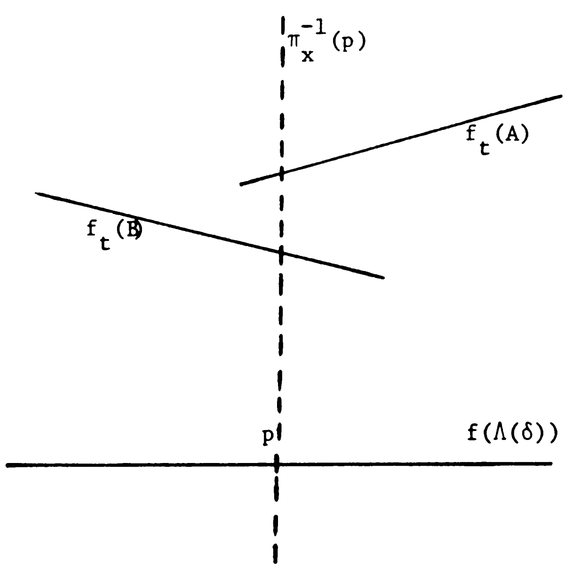

FIGURE 7.2

Claim. For all $1 \leqslant i \leqslant r$,

(i) $f_t(A_k)$ and $f_t(A_j)$ do not intersect transversally in their interiors for $k, j \leqslant i$,

(ii) D_i is convex, and

(iii) if $f_t(A_k)$ is Euclideanly-above $f_t(A_j)$ for distinct $k, j \leqslant i$, then A_k is above A_j (so that $k > j$).

Demonstration. We will proceed by induction on i. The case $i = 1$ is trivial, since it is easy to see (from the definition of the A_j's) that $f_t(A_1)$ joins $f_t(V(f))$ to a point in $\langle f(v(f)), e_R\rangle$. Now suppose the claim holds for $i-1$; we will first check that both endpoints of $f_t(A_i)$ are in $\overline{\partial D_{i-1} - \langle f_t(V(f)), e_R\rangle}$. Let b be an endpoint of A_i; it is clear from the definition of the A_i's that b is immediately above some A_m, $m < i$; hence $f_t(b) \in \overline{T_R - D_{i-1}}$. Suppose

$$f_t(b) \in T_R - D_{i-1} = \overline{T_R - D_{i-1}} - \left(\partial D_i - \langle f_t(V(f)), e_R\rangle\right);$$

either $f_t(b) \in \langle f_t(V(f)), f(v(f))\rangle$ or not. In the latter case, it is seen that $f_t(b)$ must be Euclideanly-below (in the obvious sense) some $f_t(A_k)$ $(k < i)$ which intersects ∂D_{i-1}. See Figure 7.4. b is immediately above some A_q $(\neq A_k)$, where $f(A_q)$ must be Euclideanly-below $f_t(A_k)$. However, (iii) applied to $i-1$ implies that A_k is above A_q, and since b is above A_k it follows that b could not have been immediately above A_q, a contradiction. The other case is that $f_t(b) \in \langle f_t(V(f)), f(v(f))\rangle$; since we are assuming that $f_t(b) \notin \partial D_i$, it is easy to see that $f_t(b) \in f(v(f))$, so that $b \in [f^{-1}f(v) \cap \Lambda(\delta)] - V(f)$. Since $V(f)$ contains boundary vertices on the top side of $\Lambda(\delta)$, it follows that A_1 (which intersects $V(f)$) must be above A_i. In that case, however, some subset of $A_i, A_{i-1}, \ldots, A_1$ (containing A_i and A_1) contradicts Lemma 6.1. Thus we have seen that $f_t(b) \notin T_R - D_{i-1}$, so

$$f_t(b) \in \overline{\partial D_{i-1} - \langle f_t(V(f)), f(V(f))\rangle}\,.$$

(i) and (ii) now follow for i using (i) and (ii) for $i-1$, together with the above observation, and (iii) similary follows for i using (i), (ii), (iii) for $i-1$. This proves the claim, and hence $f_t \in R(K)$ for all t.

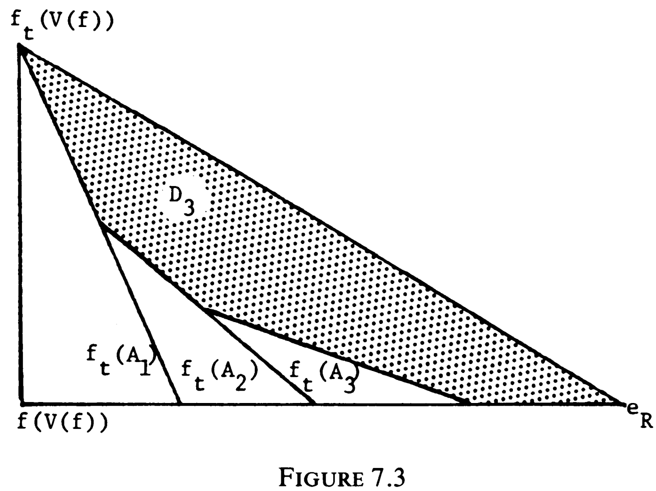

FIGURE 7.3

To see that f_t is oriented, there are three types of noncollapsed 1-simplices (with respect to f_t) for which we need to examine edge-point-inverses. If a (noncollapsed) 1-simplex is not in $\Lambda(\delta)$, then any f_t-edge-point-inverse with respect to an interior point is the same as the corresponding f-edge-point-inverse, which is an arc. If $A \in \Lambda(\delta)^1$ is not collapsed by either f_t or f, then it is seen by the construction of f_t that any f_t-edge-point-inverse with respect to an interior point of A is a submanifold of the corresponding f-edge-point-inverse, and hence is also an arc. Finally, if $A \in \Lambda(\delta)^1$ is collapsed by f but not by f_t, then A is in the piece of $\Gamma(v)$ that is pulled apart; it is easy to see from the definition of a pulling path that the f_t-edge-point-inverses of interior points of A are also arcs (see Figure 5.1), and this completes the proof of Case 1.

Case 2. *No f-segment complex has a side vertex.* As before, there must be a nontrivial f-segment complex; call it $\Lambda(\delta)$. By Lemma 3.3(i), $\Lambda(\delta)$ is simple, $\partial\Lambda(\delta) = E_1 \cup S_1 \cup E_2 \cup S_2$ for appropriately defined E_i, S_i, and in the present case each S_i is a single (noncollapsed) 1-simplex. It follows that at least one of the E_i is not a single vertex; suppose it is E_1. Let e be a vertex of E_1. Since no f-segment complex has side vertices, Lemma 4.1 implies that e satisfies hypothesis (ii) of Lemma 5.1 and $\Gamma(e)$ is defined appropriately. We will pull apart $\Gamma(e)$ just like $\Gamma(v)$ in Case 1, the only difference being that here we need to find a piece of $\Gamma(e)$ which has two teeth in different f-segment complexes; once we find such teeth, the construction of f_t and the proof that it works as desired are exactly analogous to Case 1. We find the teeth as follows.

$E_1 \subset \Gamma(e)$, so some teeth of $\Gamma(e)$ must lie in $\Lambda(\delta)$; we want to find some tooth of $\Gamma(e)$ not contained in $\Lambda(\delta)$. Note, first of all, that no 1-simplex of $\Gamma(e)$ is in ∂K, so every 1-simplex in $\Gamma(e)$ is an edge of two 2-simplices in K. Consider the pieces $R_1, \ldots, R_p$ of $\Gamma(e)$ which intersect E_1. If some R_i is a 1-simplex, then this 1-simplex is the edge of one tooth γ in $\Lambda(\delta)$ and another tooth η not in $\Lambda(\delta)$ (for if both γ and

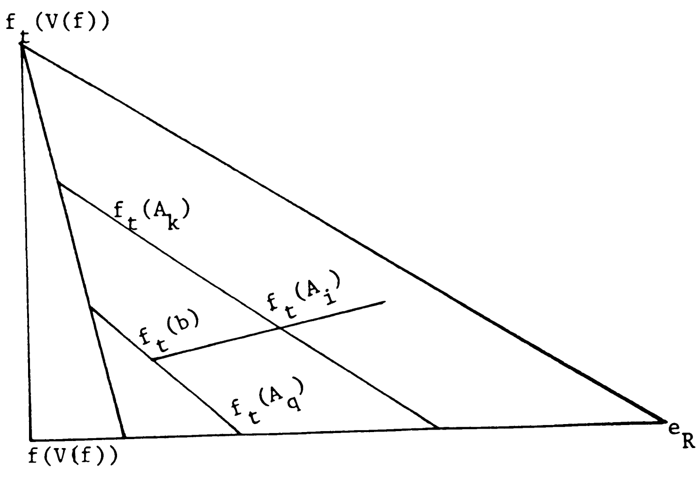

FIGURE 7.4

η were in $\Lambda(\delta)$, then $R_i \subset E_1$ could not be in $\partial\Lambda(\delta)$). If no R_i is a 1-simplex, then they are all (nontrivial) 2-disks. If all the teeth of all the R_i are in $\Lambda(\delta)$, then ∂R_i is a polygonal circle contained in $\Lambda(\delta)$ for all i. However, $\Lambda(\delta)$ is a simply connected subcomplex of K, so that each R_i must be contained in $\Lambda(\delta)$, a contradiction to the definition of $\Gamma(e)$. Thus we can find some teeth γ, η of $\Gamma(e)$ belonging to some R_i, with γ in $\Lambda(\delta)$, and η not in $\Lambda(\delta)$. This completes the proof of the proposition. □

8. Proof of Theorem 1.2. (6) ⇒ (1). The first part of (6) is exactly the same as saying f is oriented, and the second part is $f \in R(K)$, so (1) follows from Corollary 7.2.

(1) ⇒ (2) ⇒ (3). These implications are trivial.

(2) ⇒ (5). Clearly (2) implies that $f \in R(K)$; as in the proof of (3) ⇒ (6) below, it follows from (2) that $f^{-1}f(x)$ is simply connected for any $x \in K$, so in particular (5) holds.

(5) ⇒ (6). $f \in R(K)$ implies the second part of (6); to see that the first part holds assume otherwise, i.e. there is some $A \in K^1$ with a point $x \in \operatorname{int} A$ such that $f^{-1}f(x) \cap K^0 = \varnothing$ and $f^{-1}f(x)$ is not simply connected. By Lemma 2.1, $f^{-1}f(x)$ must contain a component which is a polygonal circle C. C intersects some noncollapsed 1-simplices (but no collapsed ones), all of which must lie in the same f-segment complex. Let V be the set of vertices of these 1-simplices which are outside of C, and let $v \in V$ be such that $f(v)$ is no farther from $f(x)$ than $f(w)$ for any $w \in V$. It is easy to check that $f^{-1}f(v)$ contains a polygonal circle S which is concentric with C, outside of it. Since $f(C) \neq f(S)$, $f^{-1}f(v)$ is not simply connected, a contradiction, so the first part of (6) holds.

(3) ⇒ (6). We only need to show that f is ordered, so suppose not; let $A \in K^1$ be such that there is a point $x \in \operatorname{int} A$ with $f^{-1}f(x) \cap K^0 = \varnothing$ and $f^{-1}f(x)$ not simply connected. Let v and S be as in the proof of (5) ⇒ (6), and let u be any vertex inside the region bounded by S (such u must exist). Now, any topological embedding $g\colon K \to \mathbf{R}^2$ will have the property that $g(u)$ is in the interior of the region bounded by $g(S)$; hence, since $f(S)$ is a point, it is seen that f is at least as far as $\frac{1}{2}\|f(S) - f(u)\| \geqslant \varepsilon(f)$ from any topological embedding $k \to \mathbf{R}^2$, a contradiction, so f is ordered.

(4) ⇒ (6). Since $\det(g|\delta) > 0$ in $^*\mathbf{R}$ for all $\delta \in k^2$, $\det(^\circ g|\delta) \geqslant 0$ in $\mathbf{R}$; hence $f = {}^\circ g \in R(K)$, which is the second part of (6). Now suppose the first part of (6) does not hold. Let u and S be as in the proof of (3) ⇒ (6), noting that $\|f(S) - f(u)\| > 0$ (in $\mathbf{R}$). Since g is infinitesimally close to f pointwise, it follows that

$$^\circ\big(\|g(S) - g(u)\|\big) > 0 \quad (\text{in } \mathbf{R});$$

this contradicts the fact that g is in $E(K, (^*\mathbf{R})^2)$ and $g(S)$ is an infinitesimally small circle, by applying the Transfer Principle of nonstandard analysis (see [**D**, p. 28]) to the analogous contradiction in the real case.

(6) ⇒ (4). By the proof of Corollary 7.2 we may assume f is boundary-nice, so that $f \in \mathrm{OBR}(K)$. We then construct the homotopy as in the proof of Proposition 7.1, but we only move $V(f)$ by an infinitesimally small (but nonzero) amount. Because

$E(K, (^*\mathbf{R})^2)$ is defined in terms of determinants, the proof of Proposition 7.1 also works infinitesimally, yielding the desired $g \in E(K, (^*\mathbf{R})^2)$ at the end of the homotopy. $\square$

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