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SIMPLEXWISE LINEAR NEAR-EMBEDDINGS OF A 2-DISK INTO $\mathbb{R}^2$. II

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Let $K \subset \mathbb{R}^2$ be a finitely triangulated 2-disk; a map $f\colon K \to \mathbb{R}^2$ is called *simplexwise linear* (*SL*) if $f|\sigma$ is affine linear for each (closed) 2-simplex σ of K. Let $L(K) = \{SL$ homeomorphisms $K \to K$ fixing ∂K pointwise$\}$, let $\overline{L(K)}$ denote its closure in the space of all SL maps $K \to \mathbb{R}^2$. In this paper it is proved that if K is strictly convex $\overline{L(K)}$ is 1-connected.

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simplexwise linear near embeddings
spaces of embeddings

1. Introduction

Let $K \subset \mathbb{R}^2$ be a triangulated 2-disk. A map $f\colon K \to \mathbb{R}^2$ is called *simplex linear* (SL for short) if $f|\sigma$ is an affine linear map for every simplex $\sigma \in K$. Let $L(K)$ be the space of maps

$$L(K) = \{f\colon K \to K \mid f \text{ is an SL homeomorphism fixing } \partial K \text{ pointwise}\}.$$

Each SL homeomorphism $K \to K$ is determined by its values on interior vertices, so that $L(K)$ is naturally identified with an open subset of $\mathbb{R}^{2k}$, where k is the number of interior vertices of K. Consequently, the closure $\overline{L(K)}$ of $L(K)$ is well defined. In this paper, we prove the following theorem.

Theorem. *If K is strictly convex, then $\overline{L(K)}$ is 1-connected.*

Remarks. Bing and Starbird's example of a triangulated 2-disk K for which $L(K)$ is not connected, [1, Example 4.1] also has the property that $\overline{L(K)}$ is not connected. To see this, note that the image of vertex f (in [1, Fig. 4.2]) must lie to the right of line 1 or the left of line 2 if a map is to be in $L(K)$, and hence $L(K)$

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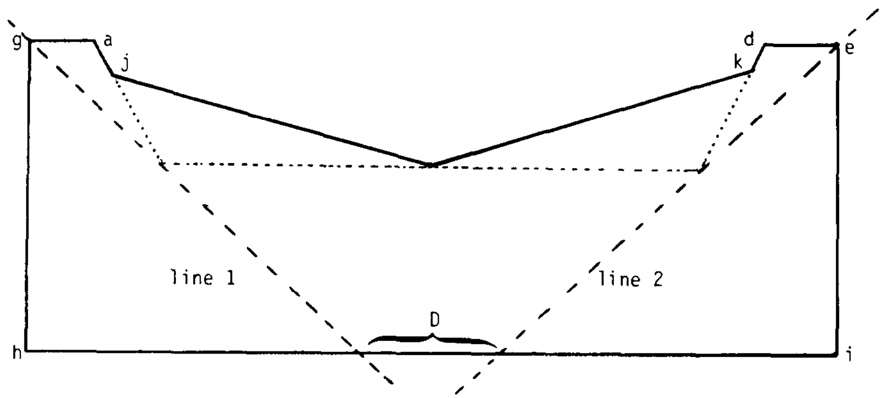

Fig. 1

has two components (or sets of components) which are at least distance D (in Fig. 1) apart; hence $\overline{L(K)}$ cannot be connected. Therefore some restriction on K is necessary to prove the theorem.

Interest in the space $\overline{L(K)}$ arose from the study of $L(K)$ and related spaces in [4] and [3] respectively, where essential use was made of $\overline{L(K)}$. Moreover, now that it is known (from [4]) that $L(K)$ is naturally identified with an open ball in $\mathbb{R}^{2k}$ if K is convex, it is of interest to see precisely how $L(K)$ sits in $\mathbb{R}^{2k}$. It is easy to find examples which show that $L(K)$ is not convex in general, even if K is convex. However, the result of this paper shows that $L(K)$ cannot lie in $\mathbb{R}^{2k}$ in too bad a way. Any further restriction found on the topology of $\overline{L(K)}$ will thus say more about how $L(K)$ sits in $\mathbb{R}^{2k}$. There is some evidence in [2] that in fact $\overline{L(K)}$ is a topological manifold with boundary equal to $\overline{L(K)} - L(K)$, which would imply that $\overline{L(K)}$ is a closed ball, but this is still unknown. For more information concerning simplexwise linear maps, see [4].

This paper is a sequel to [2], where a characterization of elements of $\overline{L(K)}$ was given. We will assume familiarity with [4, 2, 3], and will use definitions and results of these papers without restatement; in particular, we assume familiarity with the details of the proof of Proposition 7.1 of [2]. For more information concerning *SL* maps in general see [4].

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2. Preliminaries

K will always denote a triangulated 2-disk in $\mathbb{R}^2$ with k interior vertices.

Definition. Let $R(K)$ be the space of maps as defined in [2, §1], with the added requirement that all maps fix ∂K pointwise.

Remark. Theorem 1.2, and the proof of Proposition 7.1, both in [2], hold with $L(K)$, $\overline{L(K)}$ replacing $E(K)$, $\overline{E(K)}$, and with the above defined $R(K)$ replacing the $R(K)$ in [2].

Definition. Let K be as above. An interior vertex v of K is called *movable to boundary vertex* b_v if

(i) v and b_v are the endpoints of a 1-simplex of K;

(ii) if a vertex $w \in \operatorname{link}(b_v, \partial K)$ is the endpoint of a 1-simplex of K containing v, then $\langle v, b_v, w\rangle$ is a 2-simplex of K.

Remarks. (1) A vertex v may be movable to more than one boundary vertex; in that case b_v will be arbitrarily fixed throughout the proof.

(2) For any triangulated 2-disk with interior vertices, it is easy to see that there is at least one movable vertex in the interior of K.

Definition. If K is strictly convex, and v is movable to b_v, we will construct the *induced* triangulated 2-disk (or 2-disks) $K_v^1, \ldots, K_v^p$ $(1 \leq p)$ as follows. Let T be the identification space obtained from K by collapsing the 1-simplex $\langle v, b_v\rangle$ to a point, and collapsing both 2-simplices containing $\langle v, b_v\rangle$ to 1-simplices by linearly extending the collapsing of $\langle v, b_v\rangle$. See Fig. 2. Clearly T is topologically a 2-disk. The definition of movability insures that T is also triangulated; (using some incorrectly chosen v could result in double edges). It follows from the proof of Theorem 2.2 in [1] that T can be embedded in $\mathbb{R}^2$ as a rectilinearly triangulated 2-disk with strictly convex boundary. If T has no spanning 1-simplices, call this strictly convex embedding K_v^1, and this is the induced disk; if T has spanning 1-simplices, then these 1-simplices chop up the strictly convex embedding of T into the union of smaller strictly convex disks $K_v^1, \ldots, K_v^p$, and these are the induced disks. See Fig. 2.

Definition. For v movable to b_v, let

$$D(v, b_v) = \{f: K \to \mathbb{R}^2 | f \text{ is SL and } f(v) = f(b_v)\}.$$

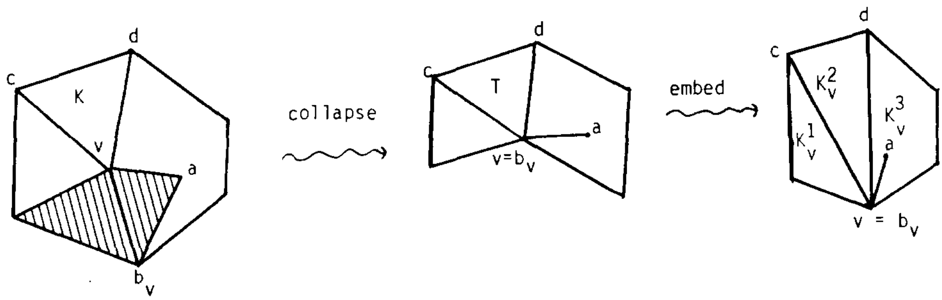

Fig. 2

Lemma 1. *For K strictly convex, with v movable to b_v, and with induced disks $K_v^1, \ldots, K_v^p$, then*

$$\overline{L(K)} \cap D(v, b_v) \approx \overline{L(K_v^1)} \times \cdots \times \overline{L(K_v^p)},$$

(where $\approx$ denotes homeomorphism).

Proof. This follows easily from the analog for $\overline{L(K)}$ of Theorem 1.2 of [2], using criterion (6). The second part of the criterion is simply being in $\overline{R(K)}$, and this holding for a map in one of $\overline{L(K)} \cap D(v, b_v)$ or $\overline{L(K_v^1)} \times \cdots \times \overline{L(K_r^p)}$ clearly implies it holds for the other. Similarly for the first part of the criterion, since the simple connectivity mentioned in the criterion is preserved when viewed in either space. □

Definition. Given any unit vector $u \in s^1$, let $\langle u \rangle$ be the line containing u, and let $\pi_u : \mathbb{R}^2 \to \langle u \rangle$ be orthogonal projection. Let $\Pi_u : \mathbb{R}^{2k} \to \mathbb{R}^k$ be given by

$$(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) \mapsto (\pi_u(x_1, y_1), \ldots, \pi_u(x_k, y_k)).$$

In the proof that $L(K)$ is an open ball for K convex, given in [4], crucial use was made of the facts that for any $f \in L(K)$: $\Pi_u^{-1}\Pi_u(f) \cap \overline{L(K)} = \Pi_u^{-1}\Pi_u(f) \cap R(K)$, the set $\Pi_u^{-1}\Pi_u(f) \cap \overline{L(K)}$ was convex, and the collection $\{\Pi_u^{-1}\Pi_u(f) \cap \overline{L(K)} | f \in L(k)\}$ decomposed the space $\Pi_u^{-1}\Pi_u(L(K)) \cap \overline{L(K)}$ continuously. If we allow f to be in $\overline{L(K)}$, then none of these statements are true. The first fact is seen to be false by example in Fig. 4 of [4], with $u = \langle f(v_1), f(v_4) \rangle$. The second fact is shown false in the following example, and the falsity of the third fact follows from the falsity of the second, since the limit of convex sets is convex.

Example 1. Let K be the triangulation given in Fig. 3 (a), with $\langle d, e \rangle$ vertical. Let u be the unit vector in the y-axis direction. We will define $f, g \in \overline{L(K)}$ by constructing them as limits of maps in $L(K)$. Let g be defined by letting the vertices in Fig. 3 (b) move to the line $\langle d, e \rangle$, and let f be defined similarly with respect to Fig. 3 (c). Assume that $f(a) = g(a) = a$, that $f(c)$ is very close to $f(a)$ and $f(b)$ is very close to $f(e)$, and that $g(c)$ is halfway between $g(a)$ and $g(d)$, with $g(b)$ very close to $g(c)$. Consider the straight line homotopy from f to g. This homotopy fixes all vertices other than c and d, and moves $f(c)$ and $f(d)$ uniformly in straight lines to $g(c)$ and $g(d)$ respectively. By the construction of f and g, it is seen that at some point in the homotopy, the image of c will cross the image of a while the image of d will still be below the image of a, and the map defined at this point of the homotopy cannot be in $\overline{L(K)}$. Since $f, g \in \Pi_u^{-1}\Pi_u(f) \cap \overline{L(K)}$, this set is not convex.

In spite of the above example, we can prove the following:

Lemma 2. *For any $f \in \overline{L(K)}$, the sets $\Pi_u^{-1}\Pi_u(f) \cap \overline{L(K)}$ and $\Pi_u^{-1}\Pi_u(f) \cap \overline{L(K)} \cap D(v, b_v)$ are contractible.*

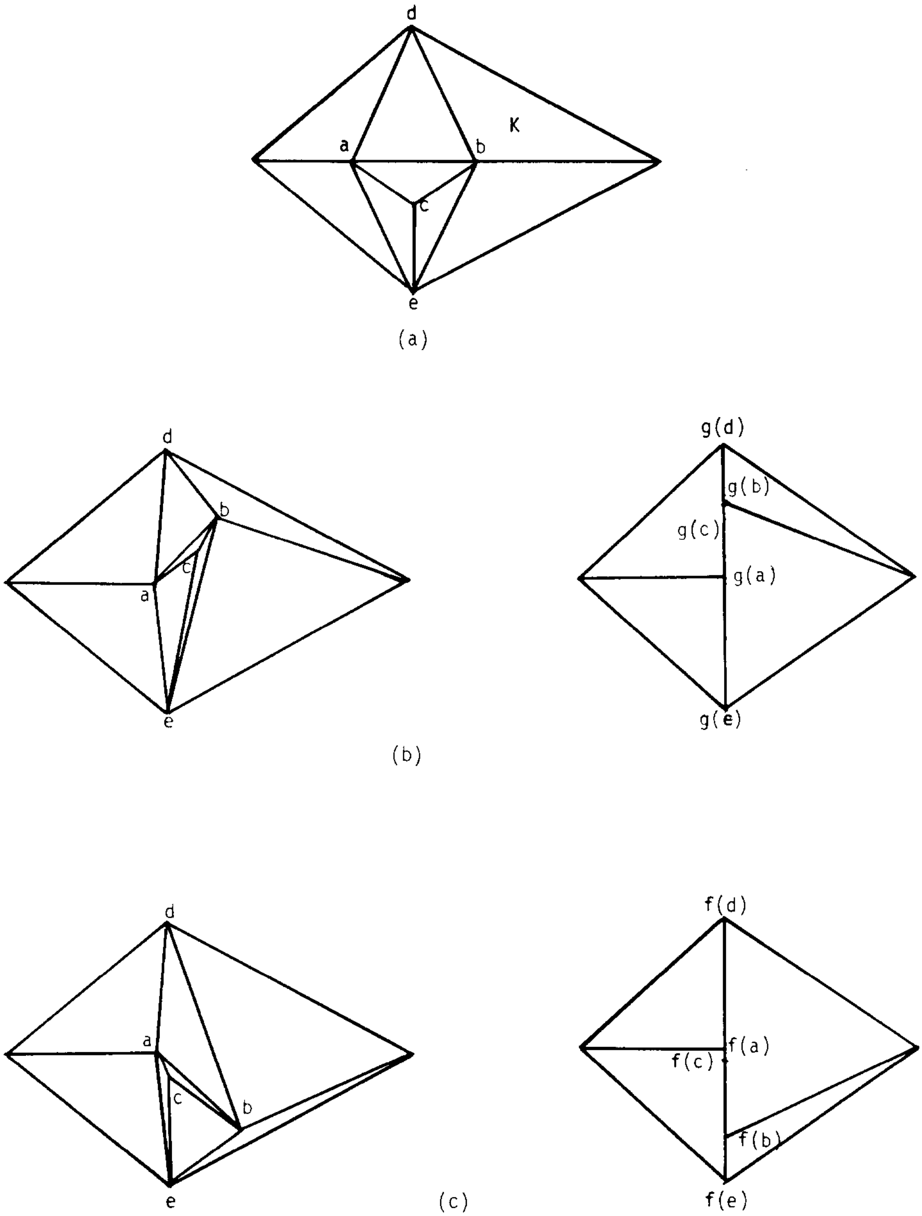

Fig. 3

Proof. First, note that $\Pi_u^{-1}\Pi_u(f) \cap R(K)$ is convex, proved just like the analogous part of Lemma 4.3 of [4]. Since intersecting with $D(v, b_v)$ is just intersecting with a codimension 2 hyperplane in $\mathbb{R}^{2k}$, it follows that $\Pi_u^{-1}\Pi_u(f) \cap R(K) \cap D(v, b_v)$ is also convex. We will prove the lemma by showing that the pair

$$(\Pi_u^{-1}\Pi_u(f) \cap R(K), \Pi_u^{-1}\Pi_u(f) \cap R(K) \cap D(v, b_v))$$

strong deformation retracts onto the pair

$$(\Pi_u^{-1}\Pi_u(f)\cap\overline{L(K)},\ \Pi_u^{-1}\Pi_u(f)\cap\overline{L(K)}\cap D(v,b_v)).$$

(Note that $\overline{L(K)}\subset R(K)$, so the latter pair is a subset of the former.) The retraction will be denoted r.

Let $g\in\Pi_u^{-1}\Pi_u(f)\cap R(K)$ be given. Note that criterion (5) of [2, Theorem 1.2] implies that $g\in\overline{L(K)}$ iff $g^{-1}g(v)$ is simply connected for every vertex v of K. For any vertex w, let B be a component of $g^{-1}g(w)$, and let $\hat{B}$ denote the union of B with the bounded components of $\mathbb{R}^2-B$. Clearly $\hat{B}$ is simply connected, and $\hat{B}=B$ precisely when B is simply connected (recall B is polyhedral). B is called *maximal* if it is not contained in some $\hat{E}$ with $E\neq B$. For any such $\hat{B}$, a simple topological degree argument, together with the fact that $g\in R(K)$, shows that $g(\hat{B})$ is the union of finitely many line segments. Moreover, $g\in\Pi_u^{-1}\Pi_u(f)$ with $f\in\overline{L(K)}$ implies that $g(\hat{B})$ actually lies in the line $\pi_u^{-1}(g(B))$, (noting that $g(B)$ is a point).

We now define $r(g)$ by specifying its values on interior vertices of K. If v is an interior vertex, then there is a unique maximal component B_v of some $g^{-1}g(w)$ with $v\in\hat{B}_v$ (w is a uniquely determined vertex, possibly v). Of course, it could happen that $\hat{B}_v=B_v=\{v\}$. Now define $r(g)(v)=g(B_v)$. The fact that each $g(\hat{B}_v)$ lies in the line $\pi_u^{-1}(g(B_v))$ implies r maps $\Pi_u^{-1}\Pi_u(f)\cap R(K)$ into itself. It is easy to check that r is actually a retraction

$$\Pi_u^{-1}\Pi_u(f)\cap R(K)\to\Pi_u^{-1}\Pi_u(f)\cap\overline{L(K)}.$$

To see that

$$r(\Pi_u^{-1}\Pi_u(f)\cap R(K)\cap D(v,b_v))=\Pi_u^{-1}\Pi_u(f)\cap\overline{L(K)}\cap D(v,b_v),$$

simply note that for $g\in\Pi_u^{-1}\Pi_u(f)\cap R(K)\cap D(v,b_v)$ we have $v\in B_v=B_{b_v}$, since $b_v\in\partial K$ and $g(v)=b_v$. Finally, since $\Pi_u^{-1}\Pi_u(f)\cap R(K)$ is convex, the straight line homotopy from r to the identity shows that r is a strong deformation retraction. □

Definition. Given two SL maps $f, g: K\to K$ fixing ∂K pointwise, we say that f and g are *similarly collapsed* if for every 2-simplex $\delta\in K$, δ is respectively not-collapsed, of type PC, of type EC (with specified isolated vertex) or of type SC (with specified middle vertex) with respect to f iff it is the same with respect to g, and if for every 1-simplex $A\in K$, $f(A)$ is a line segment or a point iff $g(A)$ is. (See [2, §2] for definitions.) For any SL map $f: K\to K^2$, let

$$D(f)=\{g:K\to K\mid g\text{ is SL, fixes }\partial K\text{ pointwise, and } f\text{ and }g\text{ are similarly collapsed}\}.$$

The sets $D(f)$ are clearly disjoint, and there are only finitely many of them.

The space $L(K)$ is a semi-algebraic subset of $\mathbb{R}^{2k}$ (for a definition of semi-algebraic, see [5, §1]). This follows from Lemma 4.1 of [4], with each C_v of $\mathscr{C}$ the singleton set $\{v\}$; (note that $L(K,\mathscr{C})$ in [4] is not our $L(K)$, whereas $E(k,\mathscr{C})$ with the above

$\mathscr{C}$ is our $L(K)$). It now follows from Proposition I of [5] that $\overline{L(K)}$ is semi-algebraic. It can be checked that for each $f \in \overline{L(K)}$, $D(f)$ is semi-algebraic. Hence, using Hironaka's Triangulation Theorem for systems of semi-algebraic sets (in [5]), we obtain the following lemma.

Lemma 3. *There is a finite triangulation of* $\overline{L(K)}$ *such that every set* $D(f)$ (*with* $f \in \overline{L(K)}$) *is the union of finitely many* (*open*) *simplices.*

3. Proof of Theorem

The fact that $\pi_0 \overline{L(K)} = 0$ follows immediately from Theorem 5.1 of [4] and Corollary 7.3 of [2]. To prove $\pi_1 \overline{L(K)} = 0$, we will proceed by induction on the number k of interior vertices of K. The result is trival if $k = 0$, so suppose $k \geq 1$. By a remark above, we can pick an interior vertex v of K which is movable to boundary vertex b_v. Given any map $C : S^1 \to \overline{L(K)}$ representing an element of $\pi_1 \overline{L(K)}$, we will homotop C to a map $\hat{C} : S^1 \to \overline{L(K)}$ which has $\hat{C}(s)(v) = b_v$ for all $s \in S^1$. By Lemma 1 $\hat{C}(S^1) \subset \overline{L(K_v^1)} \times \cdots \times \overline{L(K_v^p)}$, where $K_v^1, \ldots, K_v^p$ are the induced disks determined by v and b_v. Since each K_v^i has less than k interior vertices, the result will follow by induction.

The construction of the homotopy C to $\hat{C}$ uses the parallel track method of [3, §3]. Applying that method to our situation, we could start with any map in $L(K)$, and homotop it by moving the image of v in a straight line until it hits b_v; the method of [3] says how to move certain other interior vertices, keeping them in 'parallel tracks', so that the homotopy stays in $\overline{L(K)}$. Since K is strictly convex, no boundary vertices need be moved. Unfortunately, there is no canonical way of extending this construction to all of $\overline{L(K)}$; an example follows this proof. However, we can extend the construction continuously over small intervals in S^1, and then patch things together.

More specifically, for any $f \in L(k)$, let $P_t(f)$ be the parallel track homotopy of [3, §3] with $P_0(f) = f$, and $P_1(f)(v) = b_v$. Given any $g \in \overline{L(K)}$, we can find a sequence $h = \{h_n\} \in L(K)$ with $h_n \to g$; we can then define a homotopy $P_t^h(g)$ to be $P_t^h(g) = \lim_{n\to\infty} P_t(h_n)$. There are, however, two problems: first, $\lim_{n\to\infty} P_t(h_n)$ may not be well defined (if $\{h_n\}$ is poorly chosen); second, even if $P_t^h(g)$ is well defined there can be another sequence $e = \{e_n\}$ with $e_n \to g$, $P_t^e(g)$ well defined, and yet $P_t^h(g) \neq P_t^e(g)$. It is clear, however, that if $P_t^h(g)$ is well defined, then $P_t^h(g) \in \overline{L(k)}$, $P_0^h(g) = g$, $P_1^h(g)(v) = b_v$, and $P_t^h(g)$ moves v in a straight line to b_v, and other vertices in straight lines parallel to $\langle g(v), b_v \rangle$. To construct the homotopy C to $\hat{C}$, it would suffice to find a sequence of maps $\{C_n : S^1 \to L(K)\}$ such that $C_n \to C$, and $\lim_{n\to\infty} P_t(C_n(s))$ is well defined for all $s \in S^1$; however, it is not clear how to construct such $\{C_n\}$.

If we restrict our attention to some interval $(a, b) \in S^1$ (where we choose some orientation of S^1 so that we have well defined intervals), for which $C((a, b)) \subset D(f)$

for some $f \in \overline{L(K)}$, then we can find a sequence of maps $\{C_n : (a, b) \to L(k)\}$ converging to $C|(a, b)$ such that $P_t(C_n)$ converges to a well defined, continuous homotopy which is denoted $P_t(C|(a, b))$. Such a sequence of maps can be found as follows. By the definition of the sets $D(f)$, all maps in $C|(a, b)$ are similarly collapsed. We can now apply the pulling apart technique of the proof of Proposition 7.1 of [2] to $C|(a, b)$; this technique gives an explicit method for constructing paths in $L(K)$ which converge to any given element of $\overline{L(K)}$. In particular, it can be seen by this explicit construction that since all maps in $C((a, b))$ are similarly collapsed, we can not only construct our sequence $\{C_n|(a, b)\}$ by this method, but we can do so by the same sequence of 'pulling apart' moves for all $s \in (a, b)$. (There are many possible sequences $\{C_n|(a, b)\}$; any sequence will do as long as all the maps are constructed in the same way, for all n and all $s \in (a, b)$.) Thus, during the homotopies $P_t(C_n(s))$, for $s \in (a, b)$, all the vertices other than v which are moved will be moved by the images of the same $C_n(s)$-segment complexes (see [2, §3]), and in the same places in these images. It then follows easily that the $P_t(C_n|(a, b))$ converge to some well defined homotopy $P_t(C|(a, b))$ as desired. Moreover, it can be seen that by such a choice of $\{C_n|(a, b)\}$, the limits $\lim_{s\to a} P_t(C(s))$ and $\lim_{s\to b} P_t(C(s))$ yield well defined homotopies called $P_t(C(a))$ and $P_t(C(b))$, and $P_t(C|[a, b])$ is continuous.

The homotopy C to $\hat{C}$ is now constructed as follows. By Lemma 3, and the simplicial Approximation Theorem, we can homotop C to a simplicial map $C_1 : S^1 \to \overline{L(K)}$ for some triangulation of S^1, and the triangulation of $\overline{L(K)}$ as in the lemma. Since $\overline{L(K)}$ is finitely triangulated, S^1 is compact, and the triangulation of $\overline{L(K)}$ is compatible with the sets $D(f)$, it follows that there is a finite set of points $a_1, \ldots, a_p$ in S^1 (in increasing order with respect to the orientation of S^1) such that each $C_1((a_i, a_{i+1}))$ is contained in a single set $D(f)$, (where addition is mod p). (Note that $C([a_i, a_{i+1}])$ need not be contained in a single $D(f)$.) Next, choose small disjoint intervals $[a_i^-, a_i^+]$ containing the a_i in their interiors. Let $C_2 : S^1 \to \overline{L(K)}$ be constantly $C_1(a_i)$ on $[a_i^-, a_i^+]$, and let it be $C_1|(a_i, a_{i+1})$ on (a_i^+, a_{i+1}^-) in the obvious way; clearly C_1 is homotopic to C_2. By the previous paragraph, we can define a homotopy $P_t(C_2|[a_i^+, a_{i+1}^-])$, with $P_0(C_2(s)) = C_2(s)$ and $P_1(C_2(s))(v) = b_v$ for all $s \in [a_i^+, a_{i+1}^-]$, and for all $1 \leq i \leq p$. It remains to define the homotopies $P_t(C_2|(a_i^-, a_i^+))$. Note that $P_t(C_2(a_i^-))$ and $P_t(C_2(a_i^+))$ are (possibly different) paths in the set $\Pi_u^{-1}\Pi_u C_2(a_i) \cap \overline{L(K)}$, where u is a unit vector orthogonal to the 1-simplex $\langle v, b_v\rangle$; (recall $C_2(a_i^-) = C_2(a_i) = C_2(a_i^+)$). Both paths end in the set $\Pi_u^{-1}\Pi_u(C_2(a_i)) \cap \overline{L(K)} \cap D(v, b_v)$. By Lemma 2 this set is path connected, so we can find a path in this set from $P_1(C_2(a_i^-))$ to $P_1(C_2(a_i^+))$. This path, together with the paths $P_t(C_2(a_i^-))$, $P_t(C_2(a_i^+))$ and $C_2|[a_i^-, a_i^+]$ form a simple closed curve in $\Pi_u^{-1}\Pi_u(C_2(a_i)) \cap \overline{L(K)}$. By Lemma 2 this set is simply connected, and it follows that we can find a family of paths, parameterized by $[a_i^-, a_i^+]$, from $P_t(C_2(a_i^-))$ to $P_t(C_2(a_i^+))$ such that all paths start at $C_2(a_i)$ and end in $\Pi_u^{-1}\Pi_u(C_2(a_i)) \cap \overline{L(K)} \cap D(v, b_v)$. This family of paths gives us $P_t(C_2|(a_i^-, a_i^+))$. Putting all this together, we obtain $P_t(C_2)$ on all S^1, with $P_0(C_2) = C_2$ and $P_1(C_2) = \hat{C}$. This completes the construction of the homotopy C *to* $\hat{C}$, and hence $\pi_1 \overline{L(K)} = 0$. □

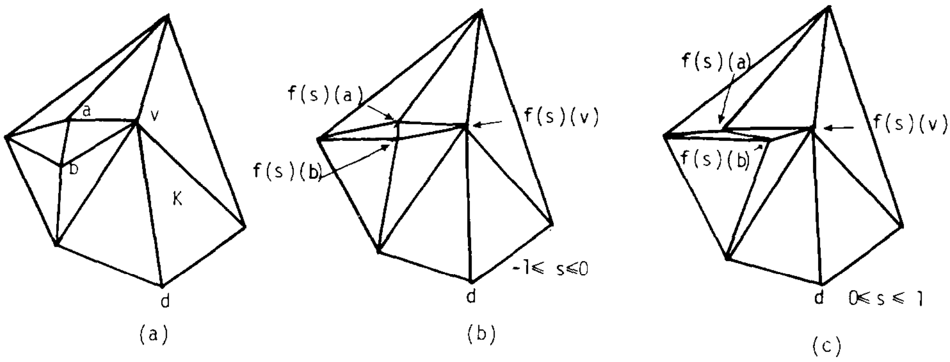

Fig. 4

Example 2. Let K be as in Fig. 4(a), with v at the origin, $a=(-1,\ ,0)$, $b=(-1,1)$ and $d=(0,-3)$. Define $f:[-1,1]\to\overline{L(K)}$ as follows: $f(s)(v)=v$ for all $s\in[-1,1]$, $f(s)(a)=(-|s|,0)$ for all $s\in[-1,1]$, and

$$f(s)(b)=\begin{cases}(-|s|,-|s|) & \text{for } s\in[-1,0],\\ (-|s|/2,-|s|^2) & \text{for } s\in[0,1].\end{cases}$$

See Fig. 4(b) and (c).

Note that $f(s)\in L(K)$ for $s\neq 0$, but that $f(0)(\langle a,b,v\rangle)$ is a point. Now, when applying the parallel tracks homotopy P_t to $f|[-1,0)$, the image of b at time $t=1$ gets closer and closer to the origin as $s\to 0^-$. On the other hand, applying P_t to $f|(0,1]$, we see that the image of b at $t=1$ gets closer and closer to approximately half way down the line segment $\langle v,d\rangle$ as $s\to 0^+$. Thus, there is no way to define $P_tf(0)(b)$ in a way that makes $P_tf(s)$ continuous at $s=0$.

Also, note that f can be approximated arbitrarily closely by maps $[-1,1]\to L(K)$ as follows. For any $0<\varepsilon<1$, let $f_\varepsilon:[-1,1]\to L(K)$ be defined by $f_\varepsilon(s)=f(s)$ for $s\in[-1,-\varepsilon]\cup[\varepsilon,1]$, and let $f_\varepsilon(s)|[-\varepsilon,\varepsilon]$ be the straight line homotopy from $f(-\varepsilon)$ to $f(\varepsilon)$. In this homotopy, all vertices except b are fixed, and the image of b is moved in a straight line.) Clearly $f_\varepsilon(s)\in L(K)$ for all $s\in[-1,1]$, and f_ε is within ε of f. We can define $P_tf_\varepsilon(s)$ for all $s\in[-1,1]$, but the above argument shows that $\lim_{\varepsilon\to 0}P_tf_\varepsilon(s)$ is not well defined at $s=0$.

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