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A COMBINATORIAL CHERN–WEIL THEOREM FOR 2-PLANE BUNDLES WITH EVEN EULER CHARACTERISTIC

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ABSTRACT

A combinatorial Chern–Weil theorem for arbitrary oriented 2-plane bundles with even Euler class over surfaces is proved. Along the way a simple method is developed to use exterior angles to calculate the curvature at the vertices of a large class of non-convex, non-immersed surfaces in $\mathbf{R}^3$.

1. Preliminaries

Various methods have been developed for calculating combinatorial characteristic classes for tangent and normal bundles of manifolds (see [HT], [Ba2], [BM], [GT] for a few examples). Recently, partly due to the interest of physicists in what they refer to as Lattice Gauge Theory, there have been efforts to work combinatorially directly with arbitrary bundles, as in [PS]. (It is true that all bundles can be dealt with via tangent and normal bundles, but it seems desirable to deal with arbitrary bundles directly.) It would be nice to have an approach based on curvature (as opposed to, say, obstruction theory), thus allowing for a combinatorial Chern–Weil type theory for arbitrary bundles. In this paper we develop such a theory for oriented 2-plane bundles with even Euler characteristic over surfaces. Our procedure has two stages: first, one passes from smooth vector bundles to a combinatorial analog of vector bundles; second, one computes Euler classes in a purely combinatorial manner from the combinatorial bundles. To emphasize that this last step is indeed combinatorial, we will do it first. (This whole procedure is analogous to

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the two stages used to compute the simplicial homology of smooth manifolds: first one triangulates the smooth manifold, and then one computes the simplicial homology of the triangulation — the latter step being the purely combinatorial one.) It is hoped that the present methods will generalize to higher dimensions and codimensions.

Partly because of a needed result, and partly for motivation, we will discuss the following topic in polyhedral geometry before proceeding to characteristic classes. There are (at least) three ways to measure the curvature at the vertices of a polyhedral surface in $\mathbf{R}^3$. The simplest way is to use the angle defect $d_v = 2\pi - \Sigma_{v\in\alpha}\,\alpha$, where the α are the angles of the triangles containing vertex v. The properties of d_v are well known (see [G1] for example). (Note that the angle defect does have a smooth analog in terms of the perimeters of geodesic disks.) The second method of finding the curvature of a polyhedral surface is the Morse-theoretic approach of [Ba1] and [Ba3], which works very nicely (and which also has a smooth analog). The disadvantage of both these methods is that they do not correspond to the usual way one thinks of curvature for smooth surfaces (e.g. Gauss' original approach); also, they are not useful for our treatment of characteristic classes. The third method for finding polyhedral curvature is by exterior angles, which is analogous to Gauss' approach to smooth surfaces. Exterior angles have been widely studied by combinatorialists for convex polyhedra in all dimensions (see [G1] and [G2] for references). In §2, 6 and 7 of this paper we give a completely elementary treatment of exterior angles that works for a reasonable class of non-convex, non-immersed surfaces in $\mathbf{R}^3$, and which seems to shed some light on the geometry of such surfaces. Hopefully this approach will generalize to higher dimensions. For those only interested in polyhedral geometry, the relevant sections (§2, 6 and 7) are independent of the rest of the paper (though the reverse is not true).

The outline of the paper is as follows. In §2 we state some definitions and results concerning polyhedral geometry of surfaces. These results are proved in §6 and 7. In §3 we discuss 2-plane bundles; in §4 we discuss combinatorial analogs of smooth vector bundles; in §5 our combinatorial Chern–Weil theorem is proved.

2. Extrinsic curvature for polyhedral surfaces

Some Notation. Let S^+ be an open hemisphere in S^2, given the usual orientation. If $x, y, z \in S^+$, let $\langle x, y\rangle$ denote the geodesic arc with x and y its endpoints, and let $\langle x, y, z\rangle$ denote the geodesic triangle with x, y and z its vertices.

DEFINITION. Given three points $x, y, z \in S^+$, let the signed angle $\alpha(x, y, z)$ be the angle (in $[-\pi, \pi]$) from $\langle x, y\rangle$ to $\langle z, y\rangle$, positive or negative depending upon whether $\langle x, z, y\rangle$ is positively or negatively oriented. Also, let Area$\langle x, z, y\rangle$ be the signed area of the triangle $\langle x, z, y\rangle$, positive or negative depending upon whether $\langle x, z, y\rangle$ is positively or negatively oriented.

DEFINITION. Let $C_n = (a_1, \ldots, a_n)$ be a triangulation of S^1, with a given orientation, and let $f\colon C_n \to S^+$ be a simplexwise geodesic (abbreviated SG) map (i.e f is a unit speed geodesic on each interval $[a_k, a_{k+1}]$). Furthermore, assume f is injective on each interval. Let $x \in S^+$ be any point. Define the area of $f(C_n)$ with respect to x to be

$$A(f, x) = \sum_{k=1}^{n} \mathrm{Area}\langle x, f(a_k), f(a_{k+1})\rangle.$$

THEOREM 2.1. *Let S^+ be an open hemisphere in S^2, and let $f\colon C_n \to S^+$ be an SG map as above. Then $A(f, x)$ is independent of the choice of $x \in S^+$.*

This theorem is proved in §6. We now apply the theorem to polyhedral surfaces. To allow for more interesting geometric situations than is possible with embedded complexes, we make the following

DEFINITION. Let $\hat{K}$ be an abstract n-dimensional simplicial complex, and let $f\colon \{\text{vertices of } \hat{K}\} \to \mathbf{R}^k$ be a map such that if $\{v_0, \ldots, v_n\}$ are the vertices of an n-simplex in $\hat{K}$, then $\{f(v_0), \ldots, f(v_n)\}$ are affinely independent in $\mathbf{R}^k$. Let K denote the collection of simplices in $\mathbf{R}^k$ with vertices $\{f(v_0), \ldots, f(v_p)\}$, for every simplex $\{\mathrm{v}_0, \ldots, \mathrm{v}_p\}$ in $\hat{K}$; we call K a *simplexwise embedded* complex in $\mathbf{R}^k$.

From now on we will restrict our attention to closed, oriented, simplexwise embedded surfaces in $\mathbf{R}^3$. In order to have a meaningful notion of extrinsic curvature, we need some local restrictions somewhat analogous to being smooth, and quite similar to the transverse fields originally considered by [C], and more recently by [L].

DEFINITION. Let $K \subset \mathbf{R}^3$ be an oriented simplexwise embedded surface. For any simplex $\eta \in K$, η^* will denote its barycenter. $K^{(i)}$ will denote the set of i-simplices of K. For each 2-simplex $\sigma \in K$, define $G(\sigma^*) \in S^2$ to be the unit normal to σ, corresponding to the orientation of σ inherited from the orientation of K. K is called *star normal at vertex* $v \in K$ if the set $\{G(\sigma^*) \mid \sigma \in K^{(2)} \cap \mathrm{star}(v, k)\}$ is contained in an open hemisphere in S^2; K is called *star normal* if it is star normal at all its vertices. If K is star normal, we then choose, for each vertex $v \in K$, some point $G(v)$ in the geodesic convex hull of the set $\{G(\sigma^*) \mid \sigma \in K^{(2)} \cap \mathrm{star}(v, K)\}$. For any such set of choices, we call the map $G: \{\sigma^* \mid \sigma \in K^{(2)}\} \cup K^{(0)} \to S^2$ a *combinatorial Gauss map* on K.

REMARKS. (1) There are some useful equivalent definitions of star normality. For example, star normality at a vertex $v \in K$ is equivalent to the existence of a vector $G(v) \in S^2$ so that for each 2-simplex $\sigma \in \mathrm{star}(v, K)$, orthogonal projection from $G(\sigma^*)$ to the line containing $G(v)$ is an orientation preserving injection. This definition is seen to be equivalent to the existence of an oriented plane P_v containing v so that for each 2-simplex $\sigma \in \mathrm{star}(v, K)$, orthogonal projection from σ to P_v is an orientation preserving injection. Finally, for any vector $x \in S^2$, let $S_x \subset S^2$ denote the open hemisphere centered at x. Then star normality at v is equivalent to the condition that

$$\bigcap \{S_{G(\sigma^*)} \mid \sigma \in K^{(2)} \cap \mathrm{star}(v, K)\} \neq \varnothing .$$

(2) If $G(v)$ is in the geodesic convex hull of the set

$$\{G(\sigma^*) \mid \sigma \in K^{(2)} \cap \mathrm{star}(v, K)\},$$

then $G(\sigma^*) \in S_{G(v)}$ for all 2-simplices σ in $\mathrm{star}(v, K)$.

(3) Not all polyhedral surfaces are star normal. For example, approximate the seam of a baseball with a polyhedral curve, and take the cone on this curve from the center of the ball. However, it is seen that a fine enough C^∞-triangulation of a smooth surface in $\mathbf{R}^3$ will be star normal (simply project onto the tangent plane).

DEFINITION. Let K be as above, let G be a choice of Gauss map, and let $v \in K$ be a vertex. Let P_v be as in Remark (1) above. Orthogonally project $\mathrm{star}(v, K)$ onto P_v. Let $\mathrm{wrap}(v, K)$ be defined to be the winding number of the projection of $\mathrm{link}(v, K)$ about the origin in P_v. (Equivalently, $\mathrm{wrap}(v, K)$ is the number of preimages in $\mathrm{star}(v, K)$ under the projection of any point sufficiently near the origin in P_v.) Let $\{\eta_1, \ldots, \eta_p\}$ be the 2-simplices of $\mathrm{star}(v, K)$ in order corresponding to the orientation of K. Choose an open hemisphere $S^+ \in S^2$ containing the set

$$\{G(v)\} \cup \{G(\sigma^*) \mid \sigma \in K^{(2)} \cap \mathrm{star}(v, K)\},$$

which exists by the definition of star normality. Let $C_p = (a_1, \ldots, a_p)$ be a triangulation of S^1, and let $f\colon C_p \to S_{G(v)}$ be the SG map given by $f(a_1) = G(\eta_1^*)$. Finally, let $A(v, K)$ be defined by $A(v, K) = A(f, G(v))$. We define the extrinsic curvature e_v at v to be

$$e_v = A(v, K) - 2\pi[\mathrm{wrap}(v, K) - 1].$$

LEMMA 2.2. *Let K be a closed, oriented, star normal, simplexwise embedded surface in $\mathbf{R}^3$, and let G be a combinatorial Gauss map on K. For all vertices $v \in K$, the quantities $A(v, K)$,* wrap(v, K) *and e_v are independent of the choice of combinatorial Gauss map.*

PROOF. It suffices to show that each of $A(v, K)$ and wrap(v, K) is independent of the choice of combinatorial Gauss map, for all vertices $v \in K$. For $A(v, K)$ this follows immediately from Theorem 2.1. For wrap(v, K), the point is to observe that the set of all possible choices of $G(v)$ is a geodesically convex subset of an open hemisphere of S^2 (by definition). Call this set Q. If $x, y \in Q$ are two choices for $G(v)$, connect them with a path in Q. It is now easy to use this path in Q to construct a homotopy of the appropriate projections of link(v, K), and the lemma follows. □

The following result, proved in §7, shows that e_v equals the angle defect referred to in §1.

THEOREM 2.3. *Let K be a closed, oriented, star normal, simplexwise embedded surface in $\mathbf{R}^3$. For each vertex $v \in K$, $e_v = 2\pi - \Sigma_{v\in\alpha}\,\alpha$, where the α are the angles of the tringles containing vertex v.*

REMARKS. (1) The need for the wrapping number in the definition of e_v is due to the existence of simplexwise embedded surfaces which "do not look like" C^∞-triangulations of smoothly immersed surfaces in $\mathbf{R}^3$. It is seen that a C^∞-triangulation K of a smooth surface in $\mathbf{R}^3$ has wrap$(v, K) = 1$ at all vertices.

(2) Theorem 2.3 shows that e_v is invariant under simplexwise linear local isometries, since that result is evident for the angle defect. (A *simplexwise linear* map (*SL* for short) is a map from a simplicial complex into $\mathbf{R}^n$ which is affine linear on each closed simplex. An SL map is a *local isometry* if it preserves the lengths of 1-simplices.) The following example shows that neither wrap(v, K) nor $A(v, K)$ alone are invariant under SL isometries.

EXAMPLE. Let B_4 (respectively B_8) be a square (resp. octagon) in the x–y plane, with sides of unit length and center at the origin. Let $\lambda =$

$(\sqrt{14-2\sqrt{2}})/2$, and let K be the suspension of B_8 from points $(0, 0, \lambda)$ and $(0, 0, -\lambda)$. Define an SL map $f: K \to \mathbf{R}^3$ by wrapping B_8 twice around B_4 (taking vertices to vertices), and having $f((0, 0, \pm\lambda)) = (0, 0, \pm 2)$. The collection of simplices $f(K)$ is a simplexwise embedded complex. The choice of λ makes f an SL local isometry $K \to f(K)$. However, it is easy to see that $\mathrm{wrap}((0, 0, \lambda), K) = 1$, whereas $\mathrm{wrap}(f(0, 0, \lambda), f(K)) = 2$. Also, by considering the angle defect d_v, it is seen that

$$A(f(0, 0, \lambda), f(K)) = A((0, 0, \lambda), K) + 2\pi.$$

Question. Are $A(v, K)$ and $\mathrm{wrap}(v, K)$ invariant under SL local isometries that are injective, or locally injective?

3. Remarks on 2-plane bundles

From this point on, assume all manifolds are closed, smooth and oriented, all maps between manifolds are smooth and orientation preserving, and all bundles are smooth and oriented. Some notation: $G_m(\mathbf{R}^k)$ is the Grassmannian of m-planes in $\mathbf{R}^k$, $\tilde{G}_m(\mathbf{R}^k)$ is the Grassmannian of oriented m-planes in $\mathbf{R}^k$, and $\gamma_m(\mathbf{R}^k)$ (resp. $\tilde{\gamma}_m(\mathbf{R}^k)$) is the canonical bundle over $G_m(\mathbf{R}^k)$ (resp. $\tilde{G}_m(\mathbf{R}^k)$). $\mathrm{Gl}(m, \mathbf{R})$ is the general linear group of real $m \times m$ matrices; $\mathrm{Gl}^+(m, \mathbf{R})$ is the subset of $\mathrm{Gl}(m, \mathbf{R})$ of matrices of positive determinant. If ξ is an m-plane bundle, let $B(\xi)$ denote the base space of ξ, and let $\chi(\xi) \in H^m(B(\xi), \mathbf{R})$ denote the Euler class of ξ.

Let ξ be a smooth (oriented) m-plane bundle. It is a standard result that ξ can be classified by maps $g: B(\xi) \to \tilde{G}_m(\mathbf{R}^k)$ for sufficiently large k; in other words, $\xi \approx g^*(\tilde{\gamma}_m(\mathbf{R}^k))$ for such maps g. In our present treatment, classifying maps are used to give bundles geometric structure, necessary for a Chern–Weil type theorem.

It follows from the definition that for an m-plane bundle ξ to be classified in $\tilde{G}_m(\mathbf{R}^{m+1})$, ξ must be isomorphic to $g^*(\tilde{\gamma}_m(\mathbf{R}^{m+1})) = g^*(TS^m)$ for some map $g: B(\xi) \to \tilde{G}_m(\mathbf{R}^{m+1}) \approx S^m$. Therefore, $w_k(\xi) = 0$ for $k \neq 0$ or m, and $p_K(\xi) = 0$ for $k \neq 0$ or $m/4$ (the latter only if $m/4$ is an integer). Whereas this is clearly a very restrictive condition in general, the following lemma gives one situation where such classifications are possible. Recall that $\tilde{G}_2(R^3) \approx S^2$.

Lemma 3.1. *Let ξ be a 2-plane bundle over a surface. The following are equivalent.*

(1) *$\chi(\xi)$ is even.*

(2) *ξ is isomorphic to the pullback of the tangent bundle of S^2 via some map $B(\xi)\to S^2$, and any two such maps are (smoothly) homotopic.*

PROOF. (1)$\Rightarrow$(2). Smooth, oriented 2-plane bundles over smooth, oriented surfaces are completely classified by their Euler classes (see [DW]; they call the Euler class the "integral w_2"). By hypothesis $\chi(\xi)=2p[B(\xi)]$ for some integer p, where $[B(\xi)]$ is the fundamental cohomology class in $H^2(B(\xi),\mathbf{R})$. For any smooth, oriented surface there exists a smooth map $g: B(\xi)\to S^2$ of degree p. Then

$$\chi(g^*(TS^2))=g^*(\chi(TS^2))=g^*(2[S^2])=2p[B(\xi)],$$

and hence $\xi\approx g^*(TS^2)$.

Suppose $g, h: B(\xi)\to S^2$ are maps such that $g^*(TS^2)\approx h^*(TS^2)$. Then

$$\begin{aligned}2(\deg(g))[B(\xi)]&=2g^*([S^2])=g^*(2[S^2])=g^*(\chi(TS^2))=\chi(g^*(TS^2))\\&=\chi(h^*(TS^2))=\cdots=2(\deg(h))[B(\xi)].\end{aligned}$$

Therefore $\deg(g)=\deg(h)$, and hence g is homotopic to h by the Hopf–Whitney Theorem [Wh, p. 244], which states that $[B(\xi),S^2]\approx H^2(B(\xi),\mathbf{R})$, where $[B(\xi),S^2]$ denotes the homotopy classes of maps $B(\xi)\to S^2$; the isomorphism is given by $g\mapsto(\deg(g))[B(\xi)]$. By smoothing theory, g is in fact smoothly homotopic to h.

(2)$\Rightarrow$(1). This is straightforward. □

REMARK. The homotopy equivalence of maps in the lemma is not entirely trivial, since in general the proof of the homotopy classification of bundles involves going into higher dimensional Grassmannians (see [H, p. 32] for example).

A definition we will need later is

DEFINITION. Let ξ be a smooth m-plane bundle, and let $g: B(\xi)\to\tilde{G}_m(\mathbf{R}^k)$ be a classifying map for ξ. If N is a manifold, and $h: N\to B(\xi)$ is a map, then the bundle $h^*(\xi)$ over N is given the *induced classifying map* $g\circ h: N\to\tilde{G}_m(\mathbf{R}^k)$. (Since $(g\circ h)^*=h^*\circ g^*$, the map $g\circ h: N\to\tilde{G}_m(\mathbf{R}^k)$ really is a classifying map for $h^*(\xi)$.)

4. Curvature and Euler class for 2-dimensional lattice gauge fields

In order to calculate characteristic classes combinatorially, we will replace a smooth bundle by a combinatorial object (analogous to the process of triangulating a smooth manifold). Such an object is called a *lattice gauge field* (LGF for short), as defined below. (The name "lattice gauge field" was coined by physicists.) A related combinatorial object we use is a *lattice classifying map* (LCM for short), also defined below. We will then define curvature, and an Euler class, in this combinatorial situation. All homology and cohomology has coefficients in $\mathbf{R}$, and is simplicial, cellular or singular depending on the situation in the obvious way.

DEFINITION. Let K be a simplicial complex. K' will denote the first barycentric subdivision of K; for any simplex $\eta \in K$, η^* will denote its barycenter. A *lattice classifying map* Θ/K into $G_m(\mathbf{R}^k)$ (called an *(m,k)-LCM* for short) is a map $\Theta : (K')^{(0)} \to G_m(\mathbf{R}^k)$ for some $k \geqq m$. An *oriented* (m,k)-LCM Θ/K is a map $\Theta : K')^{(0)} \to \tilde{G}_m(\mathbf{R}^k)$, with the added condition that for every 1-simplex $\eta = \langle \tau^*, \sigma^* \rangle \in K'$, orthogonal projection from $\Theta(\tau^*)$ to $\Theta(\sigma^*)$ is an orientation preserving injection. A *$Gl(m,\mathbf{R})$-valued lattice gauge field on K'* (called an *m-LGF* for short) Ω/K is a map $\Omega : (K')^{(1)} \to \mathrm{Gl}(m,\mathbf{R})$, and an *oriented* m-LGF is a map $\Omega : (K')^{(1)} \to \mathrm{Gl}^+(m,\mathbf{R})$. Given an oriented (m,k)-LCM Θ/K, one obtains an *induced* oriented m-LGF Ω/K as follows. For each vertex $\sigma^* \in K'$ (σ a simplex in K), choose an arbitrary orientation preserving orthogonal map $r_{\sigma^*} : \Theta(\sigma^*) \to \mathbf{R}^m$. Then, for every 1-simplex $\eta = \langle \tau^*, \sigma^* \rangle \in K'$ (where we let τ have higher dimension than σ), let $\Omega(\eta) : \mathbf{R}^m \to \mathbf{R}^m$ be defined by $\Omega(\eta) = r_{\sigma^*} \circ \Pi_{\tau,\sigma} \circ (r_{\tau^*})^{-1}$, where $\Pi_{\tau,\sigma}$ is orthogonal projection in $\mathbf{R}^k$ from $\Theta(\tau^*)$ to $\Theta(\sigma^*)$. (Another approach to constructing induced m-LGF's would be to use Riemannian parallel transport in $\tilde{G}_m(\mathbf{R}^{m+1})$.)

Two ways of obtaining m-LGF's are as follows. (1) Let K be an orientable simplexwise embedded m-manifold in $\mathbf{R}^k$. An (m,k)-LCM Θ/K is called a *tangent (m,k)-LCM* for K if it is an oriented (m,k)-LCM such that $\Theta(\tau^*)$ is parallel to τ for all simplices $\tau \in K$ (and has the same orientation as τ if τ is an m-simplex). Not every K has a tangent (m,k)-LCM (orientability is the issue here), but it is evident that suitable restrictions on K, similar in nature to the definition of star normality in §2, will insure the existence of a tangent (2,3)-LCM for a surface that is simplexwise embedded in $\mathbf{R}^3$. If K has a tangent (m,k)-LCM, one then constructs an induced *tangent m-LGF* from the (m,k)-LCM as before. We will denote tangent (m,k)-LCM's and m-LGF's by $\Theta T/K$ and $\Omega T/K$ respectively. It should be noted that if K is simplexwise embedded but not locally embedded (i.e. embedded on the star of every vertex), then the tangent (m,k)-LCM and m-LGF of K will not necessarily correspond to the tangent bundle of K.

(2) Let ξ be a smooth m-plane bundle, and suppose a classifying map $g: B(\xi) \to \tilde{G}_m(\mathbf{R}^k)$ for ξ is chosen. If $t: K \to B(\xi)$ is a C^∞-triangulation, then the assignment $\Theta(\sigma^*) = g(t(\sigma^*))$ for each simplex $\sigma \in K$ is an (m, k)-LCM, called $\Theta(\xi, g)/K$. If t is a fine enough triangulation, then $\Theta(\xi, g)/K$ will be oriented, and will thus give rise to an oriented m-LGF, called $\Omega(\xi, g)/K$. We will refer to $\Theta(\xi, g)/K$ and $\Omega(\xi, g)/K$ as being *induced* by g and t.

In order to make use of the ideas of §2, we will restrict our attention from now on to oriented (2,3)-LCM's. K will always denote a compact, oriented, triangulated surface.

DEFINITION. Let K be as above, and let Θ/K be an oriented (2,3)-LCM. If $N \in \tilde{G}_2(\mathbf{R}^3)$ is a 2-plane, let $N^\#$ be the (oriented) normal vector to N. We say Θ is *simplex-normal* if for each 2-simplex $\sigma = \langle u, v, w \rangle \in K$, there is an open hemisphere $H_\sigma \subset S^2$ such that

$$\{\Theta(\eta^*)^\# \mid \eta \in \operatorname{star}(u, K) \cup \operatorname{star}(v, K) \cup \operatorname{star}(w, K)\} \subset H_\sigma.$$

Now assume Θ/K is simplex-normal. Let $v \in K$ be a vertex. Suppose the vertices of $\operatorname{link}(v, K')$ are $\{x_1, y_1, \ldots, x_p, y_p\}$ in order corresponding to the orientation of K, where the x_k are the barycenters of the 2-simplices, and the y_k are the barycenters of 1-simplices. Choose some 2-simplex $\sigma \in \operatorname{star}(v, K)$; define an SG map $f: \operatorname{link}(v, K') \to H_\sigma$ by setting $f(z) = \Theta(z)^\#$ for each $z \in \{x_1, y_1, \ldots, x_p, y_p\}$. We define the *curvature* of Θ/K at v to be $T_v = A(f, \Theta(v)^\#)$.

Define a homology class $\hat{C}(\Theta/K) \in H_0(K)$ by letting $\hat{C}(\Theta/K)$ be represented by the simplicial 0-chain with coefficient $(1/2\pi)T_v$ at each vertex $v \in K$. Let $C(\Theta/K) \in H^2(K, \mathbf{R})$ be the Poincaré dual of $\hat{C}(\Theta/K)$ (in dual cell cohomology).

It is also possible to calculate $C(\Theta/K)$ directly from a 2-LGF Ω/K obtained from Θ/K. Whereas it is intuitively more simple to calculate $C(\Theta/K)$ as above, it is possible to give a more explicit formula using Ω/K; such a formula, and Ω/K in general, would be much easier to use on a computer than Θ/K.

First, suppose $\langle a, b, c \rangle$ is a triangle contained in an open hemisphere in S^2, with sides of length α, β and γ, and angles A, B and C. Let N_a, N_b and N_c be the oriented normal planes to a, b and c, and let $\Pi_{a,b}$, $\Pi_{a,c}$ and $\Pi_{b,c}$ be the orthogonal projections from N_a to N_b, etc. Let $r_i: N_i \to \mathbf{R}^{2'}$ be an arbitrary orientation preserving orthogonal map for $i = a$, b and c, and let

$$\Omega_{a,b} = r_b \circ \Pi_{ab} \circ (r_a)^{-1}: \mathbf{R}^2 \to \mathbf{R}^2,$$

and similarly for $\Omega_{a,c}$ and $\Omega_{b,c}$. We want to compute Area$\langle a, b, c \rangle$ using only the maps $\Omega_{a,b}$, $\Omega_{a,c}$ and $\Omega_{b,c}$.

Finding the unsigned area is straightforward. First, it is not hard to see that $\cos\alpha = \det(\Omega_{b,c})$, $\cos\beta = \det(\Omega_{a,c})$ and $\cos\gamma = \det(\Omega_{a,b})$. Using the spherical law of cosines for sides, it is then seen that

$$\begin{aligned}\cos A &= \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma} \\ &= \frac{\cos\alpha - \cos\beta\cos\gamma}{\sqrt{1-\cos^2\beta - \cos^2\gamma + \cos^2\beta\cos^2\gamma}} \\ &= \frac{\det(\Omega_{b,c}) - \det(\Omega_{a,c})\det(\Omega_{a,b})}{\sqrt{1-\det(\Omega_{a,c})^2 - \det(\Omega_{b,c})^2 + \det(\Omega_{a,c})^2\det(\Omega_{a,b})^2}},\end{aligned}$$

and similarly for the other angles of the triangle. The unsigned area of $\langle a, b, c\rangle$ is $A + B + C - \pi$, which is thus computable in terms of $\Omega_{a,b}$, $\Omega_{a,c}$ and $\Omega_{b,c}$.

The orientation of $\langle a, b, c\rangle$ is calculated as follows. Let $\{E_1, E_2\}$ be an oriented basis for $\mathbf{R}^2$, and let $P = (\Omega_{a,c})^{-1} \circ \Omega_{b,c} \circ \Omega_{a,b}$. Then $\langle a, b, c\rangle$ is positively oriented, negatively oriented or has zero area iff $P(E_1)\cdot E_2$ is positive, negative or zero respectively. To see this, note that

$$P = (\Omega_{a,c})^{-1} \circ \Omega_{b,c} \circ \Omega_{a,b} = r_a \circ (\Pi_{a,c})^{-1} \circ \Pi_{b,c} \circ \Pi_{a,b} \circ (r_a)^{-1} : \mathbf{R}^2 \to \mathbf{R}^2.$$

Therefore

$$P(E_1)\cdot E_2 = (\Pi_{a,c})^{-1} \circ \Pi_{b,c} \circ \Pi_{a,b}((r_a)^{-1}(E_1))\cdot (r_a)^{-1}(E_2),$$

since r_a is an orientation preserving orthogonal map. It is now routine to check that $(\Pi_{a,c})^{-1} \circ \Pi_{b,c} \circ \Pi_{a,b}((r_a)^{-1}(E_1))\cdot (r_a)^{-1}(E_2)$ is positive, negative or zero iff $\langle a, b, c\rangle$ has positive orientation, negative orientation or zero area respectively. Using these calculations, it is seen that there is an explicit formula for T_v at each vertex $v \in K$, using only the collection of maps in a 2-LGF Ω/K obtained from Θ/K. It should be remarked that this formula for T_v in terms of Ω/K can be applied to any 2-LGF, not just one obtained from a (2,3)-LCM; however, examples show that if a smooth 2-plane bundle is classified in $\tilde{G}_2(\mathbf{R}^4)$, then the quantities T_v computed by this formula (applied to an induced 2-LGF) will not always give the Euler class of the original bundle.

Let K be an orientable, locally embedded surface in $\mathbf{R}^3$ (in particular, K must be simplexwise embedded). Further, suppose K is suitably restricted, so that it has a tangent (2,3)-LCM $\Theta T/K$. If $v \in K$ is a vertex, it is not hard to show that if one computes T_v directly from $\Theta T/K$, then $T_v = A(v, K)$. (The issue here is that in computing T_v, one takes into account the normals to the "tangent planes" at the 1-simplices of K, which are ignored in the computation of $A(v, K)$. Whereas these normals do affect T_v in the case of arbitrary (2,3)-LCM's, they do not do so when the (2,3)-LCM is a tangent (2,3)-LCM.) If one further assumes that wrap$(v, K) = 1$, which is the case for fine enough C^∞-triangulations of smooth surfaces in $\mathbf{R}^3$, then $T_v = e_v$, which in turn equals the angle defect at v by Theorem 2.3. Hence T_v really does generalize the standard notion of curvature for polyhedral surfaces.

Definition. Let Θ_1/K and Θ_2/K be simplex-normal (2,3)-LCM's. Θ_1/K and Θ_2/K are *hemispherically related* if the same hemispheres H_σ, for all $\sigma \in K^{(2)}$, can be used in the definition of simplex-normality for both Θ_1/K and Θ_2/K.

Lemma 4.1. *Let K be a simplicial 2-complex. Suppose Θ_1/K and Θ_2/K are hemispherically related, simplex-normal (2,3)-LCM's. Then $C(\Theta_1/K) = C(\Theta_2/K)$.*

Proof. By Poincaré duality, it will suffice to show that $\hat{C}(\Theta_1/K) = \hat{C}(\Theta_2/K)$ in $H_0(K, \mathbf{R})$. For each vertex $v \in K$, let T_{1v} and T_{2v} denote the curvatures at v of Θ_1/K and Θ_2/K respectively. Let J_1 and J_2 denote the simplicial 0-chains on K with coefficients $(1/2\pi)T_{1v}$ and $(1/2\pi)T_{2v}$, respectively, at each vertex $v \in K$. We thus need to show that J_1 and J_2 are homologous. Using the compactness of K, we can find a sequence of simplex-normal (2,3)-LCM's $\Theta_1/K = \Phi_1/K = \Phi_2/K = \cdots = \Phi_n/K = \Theta_2/K$, so that any two consecutive elements of the sequence are hemispherically related, and differ on at most one vertex of K'. Hence it will suffice to prove that J_1 and J_2 are homologous if we assume that Θ_1/K and Θ_2/K differ on at most one vertex σ^* of K'. There are three possibilities, depending on the dimension of σ.

First, suppose σ is a vertex of K. Then clearly $T_{1v} = T_{2v}$ for all vertices $v \in K$ other than σ. On the other hand, the definition of curvature at σ, together with Theorem 2.1, imply that $T_{1\sigma}$ and $T_{2\sigma}$ are independent of the values of $\Theta_1/(\sigma)$ and $\Theta_2(\sigma)$, and hence are equal by the assumption on Θ_1/K and Θ_2/K.

Next, suppose $\sigma = \langle v, w\rangle$ is a 1-simplex of K. Then clearly $T_{1z} = T_{2z}$ for all vertices $z \in K$ other than v and w. Hence

$$J_1 - J_2 = (1/2\pi)(T_{1v} - T_{2v})\{v\} + (1/2\pi)(T_{1w} - T_{2w})\{w\}.$$

Let f_{1v} and f_{2v} be the maps used to compute $T_{1v} = A(f_{1v}, \Theta(v)^*)$ and $T_{2v} = A(f_{2v}, \Theta(v)^*)$, and similarly for f_{1w} and f_{2w}. The pairs of maps f_{1v} and f_{2v}, and f_{1w} and f_{2w}, each differ only at σ^*. Let η and τ be the two 2-simplices of K containing σ. Note that $f_{1v}(\eta^*) = f_{2v}(\eta^*)$, $f_{1v}(\tau^*) = f_{2v}(\tau^*)$, and similarly for w.

It is now seen that both $T_{1v} - T_{2v}$ and $T_{1w} - T_{2w}$ are equal in absolute value to the difference of the signed areas of the geodesic triangles

$$\langle f_{1v}(\eta^*), f_{1v}(\sigma^*), f_{1v}(\tau^*)\rangle \quad \text{and} \quad \langle f_{1v}(\eta^*), f_{2v}(\sigma^*), f_{1v}(\tau^*)\rangle$$

(this uses the fact that everything takes place in a hemisphere). Therefore, by taking orientations into account, it is seen that $J_1 - J_2 = \partial S$, where S is the simplicial 1-chain on K which has coefficient $(1/2\pi)(T_{1v} - T_{2v})$ at σ, and is zero elsewhere. The proof when σ is a 2-simplex is similar, where in this case S has support only on $\partial\sigma$. □

REMARK. The above lemma shows that for two (2,3)-LCM's to be "different," there needs to be some difference that is not hemispherical; this seems to correspond to the fact that smooth bundles are constructed by clutching maps, and different bundles need non-homotopic clutching maps.

5. Combinatorial Chern–Weil theorem for 2-plane bundles with even Euler characteristic

The same assumptions about bundles used in §3 will apply here. Throughout this section, let ξ be a 2-plane bundle with $\chi(\xi)$ even. (This restriction is because of Lemma 3.1.) We now apply the previous section to give a combinatorial expression for the curvature and Euler class of ξ.

DEFINITION. Let $g: B(\xi) \to \tilde{G}_2(\mathbf{R}^3)$ be a classifying map for ξ, and let $g^{\#}: B(\xi) \to S^2$ be the associated map of oriented normal vectors, i.e. $g^{\#}(x) = [g(x)]^{\#}$. A C^∞-triangulation $t: K \to B(\xi)$ is called *g-normal* if for each 2-simplex $\sigma = \langle u, v, w\rangle \in K$, $g^{\#}(\mathrm{star}(v, K) \cup \mathrm{star}(v, K) \cup \mathrm{star}(w, K))$ is contained in an open hemisphere.

REMARKS. (1) For a given classifying map g, not every C^∞-triangulation $K \to B(\xi)$ is g-normal; however, a fine enough triangulation will be.

(2) If $t: K \to B(\xi)$ is g-normal, then $\Theta(\xi, g)/K$ is simplex-normal; also, if L is any subdivision of K, then $t: L \to B(\xi)$ will be g-normal.

DEFINITION. Let ξ and g be as above, and let $t: K \to B(\xi)$ be a g-normal C^∞-triangulation. Define the cohomology class $C(\xi) \in H^2(B(\xi), \mathbf{R})$ to be $C(\xi) = (t^*)^{-1}(C(\Theta(\xi, g)/K))$.

As defined, the class $C(\xi) \in H^2(B(\xi), \mathbf{R})$ depends on (1) the choice of triangulation $t: K \to B(\xi)$, and (2) the choice of classifying map for ξ. The following lemma shows that $C(\xi)$ in fact only depends on the bundle ξ itself.

LEMMA 5.1. *Let ξ be a 2-plane bundle over a surface with $\chi(\xi)$ even. Let $g: B(\xi) \to \tilde{G}_2(\mathbf{R}^3)$ be any classifying map for ξ, and let $t: K \to M$ be a g-normal C^∞-triangulation.*

(i) *The cohomology class $(t^*)^{-1}(C(\Theta(\xi, g)/K)) \in H^2(B(\xi), \mathbf{R})$ is invariant under subdivision of K.*

(ii) *$(t^*)^{-1}(C(\Theta(\xi, g)/K))$ depends only on the bundle ξ.*

PROOF. (i) Let L be a subdivision of K. Order the vertices of K, and let $\mathrm{sd}: \{0\text{-chains of } K\} \to \{0\text{-chains of } L\}$ be as described in [F, p. 14]. The map $\mathrm{sd}_*: H_0(K) \to H_0(L)$ is an isomorphism; to prove this part of the lemma it will suffice to show that $\mathrm{sd}_*(\hat{C}(\Theta(\xi, g)/K)) = \hat{C}(\Theta(\xi, g)/L)$. If C_v denotes the curvature at vertex $v \in K$ with respect to $\Theta(\xi, g)/K$, then $\hat{C}(\Theta(\xi, g)/K)$ is represented by a 0-chain in K that is $(1/2\pi)T_v$ at v. $\mathrm{sd}_*(\hat{C}(\Theta(\xi, g)/K))$ is represented by the 0-chain in L that is $(1/2\pi)T_v$ at vertices $v \in L$ which are vertices in K, and 0 at other vertices of L. Let Φ/L be the (2,3)-LCM defined as follows. If $\tau \in L$ is a simplex, and $\eta \in K$ is the minimal dimension simplex of K containing τ, then let $\Phi(\tau^*) = \Theta(\eta^*)$, where Θ here is $\Theta(\xi, g)/K$. It is easy to check that: (1) Φ/L and $\Theta(\xi, g)/L$ are hemispherically related, and (2) $\hat{C}(\Phi/L)$ is represented by the same 0-chain as $\mathrm{sd}_*(\hat{C}(\Theta(\xi, g)/K))$. The desired result now follows from Lemma 4.1.

(ii) First, it follows immediately from (i) that $(t^*)^{-1}(C(\theta(\xi, g)/K))$ is independent of the choice of g-normal C^∞-triangulation, since any two C^∞-triangulations of $B(\xi)$ have a common C^∞-subdivision, which must be g-normal (see [W]).

Next, suppose $g, h: B(\xi) \to \tilde{G}_2(\mathbf{R}^3)$ are classifying map for ξ. Since we may subdivide our C^∞-triangulations, it is possible to choose a C^∞-triangulation $t: K \to B(\xi)$ which is both g-normal and h-normal. By Lemma 3.1, g and h are smoothly homotopic. Let $F_t: K \times [0, 1] \to B(\xi)$ be such a homotopy, i.e. $F_0 = g$ and $F_1 = h$. Subdividing K further if necessary, we may assume that for all $t \in [0, 1]$, $t: K \to B(\xi)$ is F_t-normal. By the compactness of K, there are numbers $0 = t_1 < t_2 < \cdots < t_n = 1$ such that $\Theta(\xi, F_{t_i})/K$ and $\Theta(\xi, F_{t_{i+1}})/K$ are hemispherically related for $i = 1, 2, \ldots, n-1$. Lemma 4.1 now implies that

$$C(\Theta(\xi, g)/K) = C(\Theta(\xi, F_{t_1})/K) = C(\Theta(\xi, F_{t_2})/K) = \cdots = C(\Theta(\xi, h)/K),$$

and the lemma is complete. □

LEMMA 5.2. *Let ξ be a 2-plane bundle over a surface with $\chi(\xi)$ even. Then*

(i) *if ξ has a non-zero section, then $C(\xi) = 0$;*

(ii) *if $h: N \to B(\xi)$ is a smooth map, then $C(h^*(\xi)) = h^*(C(\xi))$.*

PROOF. (i) Suppose ξ has a non-zero section. This implies that ξ is the Whitney sum of a trivial line bundle and some other bundle. Since ξ is an oriented 2-plane bundle, this other sub-bundle must also be an oriented line bundle, i.e. it is trivial. Hence ξ is trivial. Since $C(\xi)$ can be computed with any classifying map, by Lemma 5.1(ii), use a constant map $g: B(\xi) \to \tilde{G}_2(\mathbf{R}^3)$. Clearly $T_v = 0$ for all vertices $v \in K$, and hence $C(\xi) = 0$.

(ii) Let $g: B(\xi) \to \tilde{G}_2(\mathbf{R}^3)$ be a classifying map for ξ, and give $h^*(\xi)$ the induced classifying map $g \circ h$. Let $t: K \to B(\xi)$ be a g-normal C^∞-triangulation, and let $s: L \to N$ be a $(g \circ h)$-normal C^∞-triangulation. Let L_1 be a subdivision of L such that h has a simplicial approximation h_1 with respect to triangulations L_1 and K (as in [S] p. 126). Note $s: L_1 \to N$ is $(g \circ h)$-normal, as previously remarked; if K and L are chosen fine enough, then we may assume that $s: L_1 \to N$ is also $(g \circ h_1)$-normal. Since $h_1^*(\xi) \approx h^*(\xi)$, it suffices to prove the lemma for h_1. This follows from the definition of h_1^* on the (dual cell) cochain level. Let $v \in L_1$ be a vertex, and let $D(v)$ be the 2-cell dual to v. If $h_1(v) = w$, for some vertex $w \in K$, then $h_1(D(v)) \subset D(w)$, and $h_1(\partial D(v)) \subset \partial D(w)$. For each $v \in L_1$, let $\deg_v$ be the degree of the map $h_1 \mid D(v): D(v) \to D(w)$. Let J denote the dual cell 2-chain on K with value $(1/2\pi)T_v$ on $D(v)$, for each vertex $v \in K$, i.e. J represents $C(\xi)$. Then $h_1^*(C(\xi))$ is represented by $h_1^\#(J)$, which is given by

$$h_1^\#(J)(D(v)) = (\deg_v) J(D(H_1(v))), \qquad \text{for } v \in L_1.$$

However, it is not hard to see that for $v \in L_1$, the definition of T_v with respect to $h_1^*(\xi)$ implies that $T_v = (\deg_v) T_{h_1(v)}$, and the lemma now follows. $\square$

THEOREM 5.3. *If ξ is a 2-plane bundle over a surface with $\chi(\xi)$ even, then $C(\xi) = \chi(\xi)$.*

PROOF. By combining Theorem I. 11-30 with the proofs of Corollaries V.13-24 and V.13-25, all in [Sp], we obtain the following fact: if to every smooth, oriented m-plane bundle ξ over a smooth, oriented, closed manifold, m even, there is associated a cohomology class $E(\xi) \in H^m(B(\xi), \mathbf{R})$ such that $E(\xi)$ commutes with pull-backs, and such that $E(\xi) = 0$ whenever ξ has a non-zero section, then $E(\xi) = A_m \chi(\xi)$ for some constant A_m that only depends on the dimension m. The proof works for each dimension separately. Therefore, using Lemma 5.2, $C(\xi) = A_2 \chi(\xi)$ for some constant A_2 that does not depend on ξ.

It remains to show that $A_2 = 1$; computing one example will suffice. Let $TS^2 \to S^2$ be the tangent bundle of the unit 2-sphere in $\mathbf{R}^3$. Note that

$C(TS^2)([S^2]) = 2$. TS^2 is classified by the identity map $i : S^2 \to S^2 \approx \tilde{G}_2(\mathbf{R}^3)$. Let K be a regular tetrahedron centered at the origin in $\mathbf{R}^3$, and with vertices on S^2. Then radial projection $t : K \to S^2$ is an i-normal C^∞-triangulation. By convexity, it is easy to see that

$$\sum_{v \in K} T_v = 4\pi \quad (= \text{the area of } S^2),$$

and hence

$$C(TS^2)([S^2]) = \sum_{v \in K} (1/2\pi) T_v = 2.$$

Since $\chi(TS^2)([S^2]) = \chi(S^2) = 2$, $A_2 = 1$. □

6. Area for spherical cones — proof of Theorem 2.1

If C is a convex geodesic polygon in a hemisphere of S^2, with angles γ_k at its vertices ($k = 1, \ldots, n$), then the area bounded by the polygon is known to be $\Sigma_{k=1}^n \gamma_k - (n-2)\pi$. The main step in the proof of Theorem 2.1 is a generalization of this formula to the case where C is not necessarily convex or immersed. We will use the notation of §2.

Definition. Let $C_n = (a_1, \ldots, a_n)$ be a triangulation of S^1, with a given orientation, and let $f : C_n \to S^+$ be an SG map (where S^+ is an open hemisphere in S^2). Furthermore, assume f is injective on each 1-simplex of C_n. Define $S_f \subset S^+$ to be the set

$$S_f = S^+ - \bigcup_{k=1}^{n} \{\text{great circle containing } f((a_k, a_{k+1}))\},$$

where addition is mod(n). Let k be in $\{1, \ldots, n\}$. To each $x \in S_f$ we associate a quantity $\beta_k(f, x)$ as follows. The geodesics containing $\langle f(a_{k-1}), f(a_k)\rangle$ and $\langle f(a_k), f(a_{k+1})\rangle$ divide S^+ into four regions (two of which might be degenerate). We label the regions I, II, III, IV as in Fig. 6.1; the two cases in the figure correspond to whether $\langle f(a_{k+1}), f(a_{k-1}), f(a_k)\rangle$ has positive or negative orientation.

$\beta_k(f, x)$ is now defined as follows, where the orientation used is that of $\langle f(a_{k+1}), f(a_{k-1}), f(a_k)\rangle$:

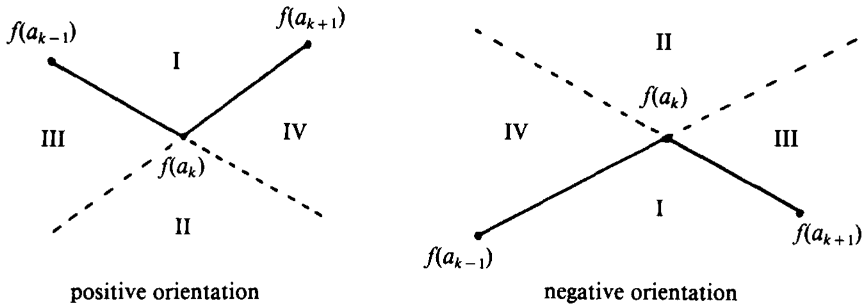

Fig. 6.1.

$$\beta_k(f,x)=\begin{cases}\alpha(f(a_{k+1}),f(a_k),f(a_{k-1}))-\pi & x\in \mathrm{I}\cup\mathrm{II},\text{ pos. orientation,}\\ \alpha(f(a_{k+1}),f(a_k),f(a_{k-1}))+\pi & x\in \mathrm{I}\cup\mathrm{II},\text{ neg. orientation,}\\ \alpha(f(a_{k+1}),f(a_k),f(a_{k-1})) & x\in \mathrm{III}\cup\mathrm{IV},\text{ any orientation.}\end{cases}$$

Finally, for $x\in S_f$, let $w(f,x)$ denote the winding number of f about x.

THEOREM 6.1. *Let $f: C_n\to S^+$ be an SG map (with notation as above), Assume f is injective on each 1-simplex in C_n; let $x\in S_f$. Then*

$$A(f,x)=\sum_{k=1}^{n}\beta_k(f,x)+2\pi w(f,x).$$

PROOF. For each $k\in\{1,\dots,n\}$ let $S_k=\pm 1$ depending on whether $\langle x,f(a_k),f(a_{k+1})\rangle$ has positive or negative orientation. Suppose $\langle x,f(a_k),f(a_{k+1})\rangle$ has (postive) angles δ_k at $f(a_k)$, ε_k at $f(a_{k+1})$, and η_k at x. Then

$$\begin{aligned}\mathrm{Area}\langle x,f(a_k),f(a_{k+1})\rangle&=S_k\{\delta_k+\varepsilon_k+\eta_k-\pi)\\&=S_k\{(\delta_k-(\pi/2))+(\varepsilon_k-(\pi/2))+\eta_k\}.\end{aligned}$$

It is easy to see that $\Sigma_{k=1}^{n}S_k\eta_k=2\pi w(f,x)$. Next, consider the vertices $f(a_k)$; it is claimed that $\beta_k(f,x)=S_k(\delta_k-(\pi/2))+S_{k-1}(\varepsilon_{k-1}-(\pi/2))$. Assuming the claim is true, the theorem follows because

$$\begin{aligned}A(f,x)&=\sum_{k=1}^{n}\mathrm{Area}\langle x,f(a_k),f(a_{k+1})\rangle\\&=\sum_{k=1}^{n}S_k\{(\delta_k-\pi/2)+(\varepsilon_k-\pi/2)+\eta_k\}\end{aligned}$$

$$= \sum_{k=1}^{n} \{S_k(\delta_k - \pi/2) + S_{k-1}(\varepsilon_{k-1} - \pi/2)\} + \sum_{k=1}^{n} S_k \eta_k$$

$$= \sum_{k=1}^{n} \beta_k(f, x) + 2\pi w(f, x).$$

It remains to prove the claim made above. Let $k \in \{1, \ldots, n\}$ be fixed. There are four cases, depending on whether each of S_{k-1} and S_k are 1 or -1.

Case 1. $S_{k-1}, S_k = 1$. There are two possibilities: either x is in region I and $\langle f(a_{k+1}), f(a_{k-1}), f(a_k) \rangle$ is positively oriented, or x is in region II and $\langle f(a_{k+1}), f(a_{k-1}), f(a_k) \rangle$ is negatively oriented. In the former case, it is seen that $\alpha(f(a_{k+1}), f(a_k), f(a_{k-1})) = \delta_k + \varepsilon_{k-1}$ (see Fig. 6.2(i)). Hence

$$\begin{aligned} \mathrm{S_k}(\delta_k - \pi/2) + S_{k-1}(\varepsilon_{k-1} - \pi/2) &= \delta_k + \varepsilon_{k-1} - \pi \\ &= \alpha(f(a_{k+1}), f(a_k), f(a_{k-1})) - \pi \\ &= \beta_k(f, x), \end{aligned}$$

by the definition of $\beta_k(f, x)$ in this case. If x is in region II and $\langle f(a_{k+1}), f(a_{k-1}), f(a_k) \rangle$ is negatively oriented, then

$$\alpha(f(a_{k+1}), f(a_k), f(a_{k-1})) = -(2\pi - (\delta_k + \varepsilon_{k-1})) = \delta_k + \varepsilon_{k-1} - 2\pi,$$

the negative sign coming from the fact that we are using signed angles (see Fig. 6.2(ii)). Hence

$$\begin{aligned} S_k(\delta_k - \pi/2) + S_{k-1}(\varepsilon_{k-1} - \pi/2) = \delta_k + \varepsilon_{k-1} - \pi &= (\delta_k + \varepsilon_{k-1} - 2\pi) + \pi \\ &= \alpha(f(a_{k+1}), f(a_k), f(a_{k-1})) + \pi = \beta_k(f, x), \end{aligned}$$

by the definition of $\beta_k(f, x)$ is this case.

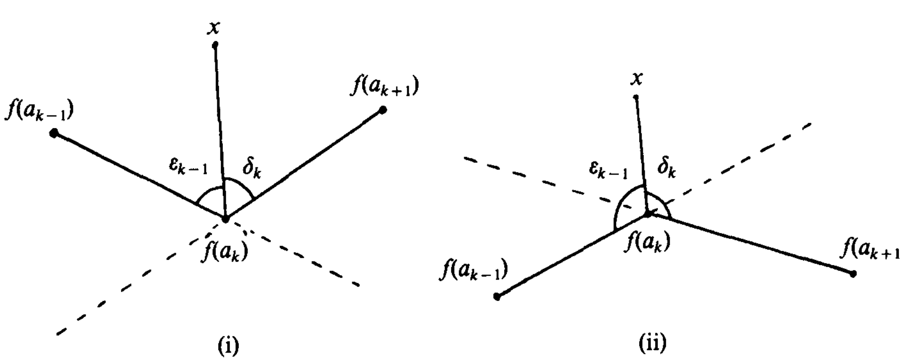

Fig. 6.2.

Case 2. $S_{k-1}, S_k = -1$. This case is similar to the previous case.

Case 3. $S_{k-1} = 1$, $S_k = -1$. Point x must be in region IV, though $\langle f(a_{k+1}), f(a_{k-1}), f(a_k)\rangle$ could be either positively or negatively oriented. In the former case, it is seen that $\alpha(f(a_{k+1}), f(a_k), f(a_{k-1})) = \varepsilon_{k-1} - \delta_K$ (see Fig. 6.3(i)). If $\langle f(a_{k+1}), f(a_{k-1}), f(a_k)\rangle$ is negatively oriented, then it still holds that $\alpha(f(a_{k+1}), f(a_k), f(a_{k-1})) = \varepsilon_{k-1} - \delta_k$ (see Fig. 6.3(ii)). It is now seen that

$$S_k(\delta_k - (\pi/2)) + S_{k-1}(\varepsilon_{k-1} - (\pi/2)) = \beta_k(f, x)$$

similarly to the previous cases.

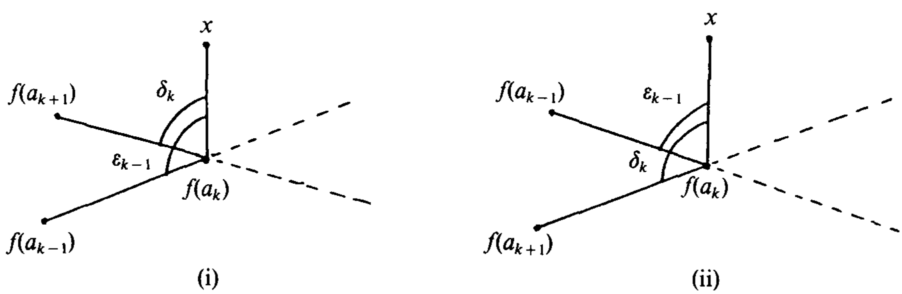

Fig. 6.3.

Case 4. $S_{k-1} = -1$, $S_k = 1$. This case is similar to the previous case. □

We note that the formula in Theorem 6.1 reduces to the standard formula $\Sigma_{k=1}^{n} \gamma_k - (n-2)\pi$ whenever $f(C_n)$ is a convex, embedded curve. If $f(C_n)$ is a convex, embedded curve, then it bounds a well defined region in S^+; choose any point x in this region. It is easy to see that (1) $w(f, x) = 1$, and (2) for each $k \in \{1, \ldots, n\}$, x is in region I, and $\langle f(a_{k+1}), f(a_{k-1}), f(a_k)\rangle$ has positive orientation. Hence

$$\beta_k(f, x) = \alpha(f(a_{k+1}), f(a_k), f(a_{k-1})) - \pi = \gamma_k - \pi.$$

It now follows that

$$\sum_{k=1}^{n} \beta_k(f, x) + 2\pi w(f, x) = \sum_{k=1}^{n} (\gamma_k - \pi) + 2\pi = \sum_{k=1}^{n} \gamma_k - (n-2)\pi$$

as desired.

Proof of Theorem 2.1. We may assume WLOG that f is injective on each interval of C_n. (If f is not injective on some interval, then it maps that interval to a point, and hence the cone from x to the image of this interval has zero area in S^+; thus $f: C_n \to S^+$ can be replaced by the obvious map $f': C_p \to S^+$, for some $p < n$.) S_f is the disjoint union of (geodesically convex) components. It is easy to see that the functions $\beta_k(f, x)$ and $w(f, x)$ are constant on each of the components of S_f (although they are not necessarily constant on all of S_f). Theorem 6.1 then implies that $A(f, x)$ is constant on each of these components. However, $A(f, x)$ is defined and continuous on all of S^+ (this uses the fact that everything takes place in an open hemisphere). It now follows easily that $A(f, x)$ is constant for all x in S^+. □

7. Proof of Theorem 2.3

Before giving the proof of Theorem 2.3, we need the following preliminaries. As in §2, assume K is a star normal simplexwise embedded surface in $\mathbf{R}^3$, and G is a combinatorial Gauss map on K. For each vertex $v \in K$, let $f: C_p \to S_{G(v)}$ be as in the definition of e_v (in §2).

DEFINITION. Let $v \in K$ be a vertex. Label the 2-simplices in star(v, K) by $\eta_1, \eta_2, \ldots, \eta_p$, in the order corresponding to the orientation of K. Assume v is at the origin of $\mathbf{R}^3$, and extend the η_k until they intersect S^2. Call the arcs of intersection g_k. We say g_k and g_{k+1} *intersect positively* (respectively *negatively*) if the angle from g_{k+1} to g_k centered at their intersection is positively (resp. negatively) oriented in S^2. For each $k \in \{1, \ldots, p\}$, we say *v is of type 1, 2, 3 or 4* at k as follows:

Type 1: g_{k-1} and g_k intersect negatively; g_k and g_{k+1} intersect negatively.
Type 2: g_{k-1} and g_k intersect positively; g_k and g_{k+1} intersect positively.
Type 3: g_{k-1} and g_k intersect negatively; g_k and g_{k+1} intersect positively.
Type 4: g_{k-1} and g_k intersect positively; g_k and g_{k+1} intersect negatively.

See Fig. 7.1. As in §2, let $S_x \subset S^2$ denote the open hemisphere centered at x. Now, fixing k, recall that the geodesics containing $\langle G(\eta^*_{k-1}), G(\eta^*_k)\rangle$ and $\langle G(\eta^*_k), G(\eta^*_{k+1})\rangle$ divide $S_{G(v)}$ into four regions (two of which might be degenerate), called regions I, II, III, IV. We say v is *well-matched at k* if v is either of type 1 or 2 at k, and $G(v)$ is in regions I or II, or if v is of type 3 or 4 at k, and $G(v)$ is in regions III or IV; otherwise we say v is *missmatched at k*. Finally, we define the local error term ε_k to be

$$\varepsilon_k = \begin{cases} 0 & \text{if } v \text{ is well-matched at } k, \\ \pi & \text{if } v \text{ is missmatched at } k. \end{cases}$$

The global error term is

$$\varepsilon(v, G) = \frac{1}{2\pi} \sum_{k=1}^{p} \varepsilon_k = \tfrac{1}{2}\{\text{number of missmatches in star}(v, K)\}.$$

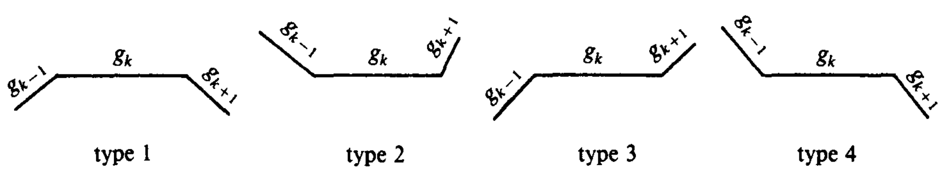

Fig. 7.1.

Lemma 7.1. *Let K be a star normal simplexwise embedded surface in $\mathbf{R}^3$, and let G be a combinatorial Gauss map. Let $v \in K$ be a vertex, let $f: C_p \to S_{G(v)}$ be as above. Choose $k \in \{1, \ldots, p\}$, and suppose η_k has angle α_k at v.*

(i) *If v is of type* 1 *or* 2 *at k, then* (1) $\langle f(\eta^*_{k+1}), f(\eta^*_{k-1}), f(\eta^*_k)\rangle$ *is positively oriented,* (2) *a point $x \in S^2$ is in regions I or II iff the plane through the origin orthogonal to x does not interesect g_k, and* (3) $\alpha(f(\eta^*_{k+1}), f(\eta^*_k), f(\eta^*_{k-1})) = \pi - \alpha_k$.

(ii) *If v is of type* 3 *or* 4 *at k, then* (1) $\langle f(\eta^*_{k+1}), f(\eta^*_{k-1}), f(\eta^*_k)\rangle$ *is negatively oriented,* (2) *a point $x \in S^2$ is in regions III or IV iff the plane through the origin orthogonal to x does not intersect g_k, and* (3) $\alpha(f(\eta^*_{k+1}), f(\eta^*_k), f(\eta^*_{k-1})) = -\alpha_k$.

(iii) *Define $\beta_k(f, G(v))$ as in* §6. *Then $\beta_k(f, G(v)) = -\alpha_k + \varepsilon_k$ (no matter what type v is at k).*

Proof. We will consider the case where v is of type 1 at k; all other cases are similar. The lemma is trivial if everything is written out in spherical coordinates (ρ, θ, ϕ). WLOG, assume that v is at the origin, and that g_k is in the x–y plane, and has endpoints $a = (1, -\alpha_k/2, \pi/2)$ and $b = (1, \alpha_k/2, \pi/2)$. Suppose that the angle between g_{k-1} and g_k is λ, and the angle between g_k and g_{k+1} is μ. It is seen that $f(\eta^*_k) = G(\eta^*_k) = (1, 0, 0)$,

$$f(\eta^*_{k-1}) = G(\eta^*_{k-1}) = (1, -(\alpha_k + \pi)/2, \lambda),$$

and

$$f(\eta^*_{k-1}) = G(\eta^*_{k-1}) = (1, -(\alpha_k + \pi)/2, \lambda),$$

the results in (i) now follow easily.

To show (iii) in case v is of type 1 at k, note that the definition of $\beta_k(f, G(v))$ depends on which region $G(v)$ is located in with respect to the geodesics containing $\langle f(\eta^*_{k-1}), f(\eta^*_k)\rangle$ and $\langle f(\eta^*_k), f(\eta^*_{k+1})\rangle$, and on the orientation of $\langle f(\eta^*_{k+1}), f(\eta^*_{k-1}), f(\eta^*_k)\rangle$. We have just seen that this orientation is positive, and that

$$\alpha(f(\eta^*_{k+1}), f(\eta^*_k), f(\eta^*_{k-1})) = \pi - \alpha_k.$$

If $G(v)$ is in region I or II, then $\varepsilon_k = 0$ (since v is well-matched in this case), and thus by definition

$$\begin{aligned}\beta_k(f, G(v)) &= \alpha(f(\eta^*_{k+1}), f(\eta^*_k), f(\eta^*_{k-1})) - \pi = (\pi - \alpha_k) - \pi \\ &= -\alpha_k + 0 = -\alpha_k + \varepsilon_k.\end{aligned}$$

If $G(v)$ is in region III or IV, then $\varepsilon_k = \pi$ (since v is missmatched in this case), and thus by definition

$$\beta_k(f, G(v)) = \alpha(f(\eta^*_{k+1}), f(\eta^*_k), f(\eta^*_{k-1})) = \pi - \alpha_k = -\alpha_k + \varepsilon_k. \qquad \square$$

Lemma 7.2. *Let K be a star normal simplexwise embedded surface in $\mathbf{R}^3$, and let G be a combinatorial Gauss map. Let $v \in K$ be a vertex, and let $f\colon C_p \to S_{G(v)}$ be as above. Then* $\mathrm{wrap}(v, K) = w(f, G(v)) + \varepsilon(v, G)$.

Proof. Assume WLOG that v is the origin of $\mathbf{R}^2$, and $G(v)$ is the North pole of S^2. Consider the geodesic polygon $C = g_1 \cup g_2 \cup \cdots \cup g_p$. Label the vertices of this polygon $(a_1, a_2, \ldots, a_p\}$, so that $g_k = \langle a_k, a_{k+1}\rangle$. Note that the map $f\colon C_p \to S_{G(v)}$ is determined by C and the point $G(v)$, and hence so are $\mathrm{wrap}(v, K)$, $w(f, G(v))$ and $\varepsilon(v, G)$. Now, we deform C by pushing its vertices "down," staying on or near their original longitudes, so that they all end up in the open Southern Hemisphere, and during the process the g_k maintain their orientations with respect to the North Pole. We can chose this homotopy to be the composition of a sequence of smaller homotopies, such that in each of these smaller homotopies only one vertex a_k moves, and one of the following three cases occurs: (1) a single arc$\langle f(\eta^*_{k-1}), f(\eta^*_k)\rangle$ passes through $G(v)$, but all pairs g_k and g_{k+1} maintain the positivity or negativity of their intersection, (2) no arc$\langle f(\eta^*_{k-1}), f(\eta^*_k)\rangle$ passes through $G(v)$, but the pair g_k and g_{k+1} which intersect in the moving vertex change the positivity or negativity of their intersection, though all other pairs g_k and g_{k+1} maintain the positivity or negativity of their intersection, or (3) no arc$\langle f(\eta^*_{k-1}), f(\eta^*_k)\rangle$ passes through $G(v)$, and all pairs g_k and g_{k+1} maintain the positivity or negativity of their intersection. Such a homotopy will not change $\mathrm{wrap}(v, K)$. $w(f, G(v))$ and $\varepsilon(v, G)$ certainly may change during the homotopy. We will indicate in the next paragraph that whereas $w(f, G(v))$ and $\varepsilon(v, G)$ may each change, the sum $w(f, G(v)) + \varepsilon(v, G)$ remains constant. To conclude the proof, it remains to note that if C is entirely in the open Southern Hemisphere, then $\mathrm{wrap}(v, K) = w(f, G(v))$, and $\varepsilon(v, G) = 0$. The former claim is straightforward, and the latter follows from part (2) in Lemma 7.1(i) and (ii), which show that $\varepsilon_k = 0$ for all k.

We need to show that the sum $w(f, G(v)) + \varepsilon(v, G)$ remains constant during any of the three types of smaller homotopies described in the previous paragraph. Clearly neither $w(f, G(v))$ nor $\varepsilon(v, G)$ changes during a type (3) homotopy. In a type (2) homotopy, it is evident that $w(f, G(v))$ is constant. We omit the details here, but it is not hard to show that all the ε_k remain constant as well; the point is that for both arcs g_k and g_{k+1} containing the moving vertex, the type of v changes from 1 or 2 to 3 or 4 (or *vice versa*), and the corresponding region in which $g(v)$ is located changes from I or II to III or IV (or *vice versa*), thus not changing whether v is well-matched or missmatched at k and $k+1$. Finally, for type (1) homotopies, we will show that when a geodesic containing an arc $\langle f(\eta_{k-1}^*), f(\eta_k^*)\rangle$ passes through $G(v)$, the sum $w(f, G(v)) + \varepsilon(v, G)$ remains constant. Call this geodesic t. By budging the homotopy slightly, we may assume that when $G(v)$ crosses t, it does not do so at $f(\eta_{k-1}^*)$ or $f(\eta_k^*)$. v could be of any type at $k-1$ and k; there are four cases, depending on whether v is of type 1 or 2, or of type 3 or 4, at each of $k-1$ and k. We will examine the case where v is of type 1 or 2 at $k-1$, and is of type 3 or 4 at k; the other three cases are similar. It is easier to think of $G(v)$ crossing t, rather than *vice versa*. There are generically three types of crossings, labeled as a, b and c in Fig. 7.2. In situations a and c, $G(v)$ does not cross the image of f, so that $w(f, G(v))$ is unchanged. In situation a, $G(v)$ moves from region III to region II at $k-1$ (so ε_{k-1} decreases by π, being of type 1 or 2), and it moves from region IV to region I at k (so ε_k increases by π, being of type 3 or 4). Thus

$$\varepsilon(v, G) = \frac{1}{2\pi} \sum_{k=1}^{p} \varepsilon_k$$

remains unchanged in situation a. Similarly in situation c. Hence, in both situations a and c, $w(f, G(v)) + \varepsilon(v, G)$ remains unchanged. In situation b, $G(v)$ crosses $\langle f(\eta_{k-1}^*), f(\eta_k^*)\rangle$ so that $\langle G(v), f(\eta_{k-1}^*), f(\eta_k^*)\rangle$ changes from being positively oriented to being negatively oriented (the other direction of crossing is similar). Hence $w(f, G(v))$ decreases by 2π. However, $G(v)$ moves from region I to region IV at $k-1$ (so ε_{k-1} increases by π, being of type 1 or 2), and it moves from region IV to region I at k (so ε_k increases by π, being of type 3 or 4). Thus $\varepsilon(v, G)$ increases by 2π in situation b, and hence $w(f, G(v)) + \varepsilon(v, G)$ remains unchanged. □

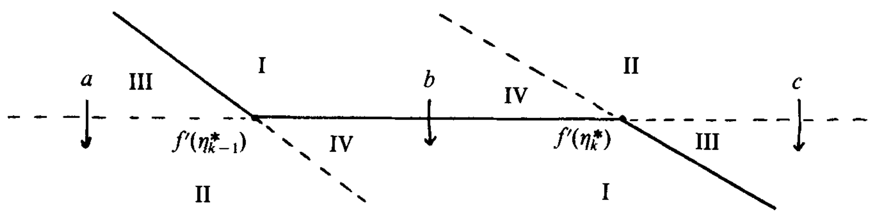

Fig. 7.2.

Proof of Theorem 2.3. Let $\{\eta_1, \ldots, \eta_p\}$ be the 2-simplices of star(v, K) in order corresponding to the orientation of K. To simplify our situation, we may assume that f is injective on each interval of C_p. Suppose conversely that $f(\eta_k^*) = f(\eta_{k+1}^*)$ for some k. Then the adjacent 2-simplices η_k and η_{k+1} are parallel. We could then modify star(v, K) by replacing η_k and η_{k+1} by a single 2-simplex in the obvious way (such a 2-simplex might not "fit" in K anymore, but the curvature e_v only depends on star(v, K)). Continuing in this way, we can eliminate all cases where f is not injective on an interval. It is not hard to see that this process would change neither $A(v, K)$ nor wrap(v, K). Hence we may assume WLOG that f is already injective on each interval.

Let α_k be the angle at v in 2-simplex η_k. For each $k \in \{1, \ldots, p\}$, we can define $\beta_k(f, G(v))$ as in §6. Then

$$\begin{aligned}
\varepsilon_v &= A(v, K) - 2\pi[\operatorname{wrap}(v, K) - 1] \\
&= \left\{\sum_{k=1}^{n} \beta_k(f, G(v)) + 2\pi w(f, G(v))\right\} - 2\pi\{w(f, G(v)) + \varepsilon(v, G) - 1\} \\
&\qquad\qquad \text{by Theorem 6.1 and Lemma 7.2} \\
&= \sum_{k=1}^{n} (-\alpha_k + \varepsilon_k) - 2\pi\varepsilon(v, G) + 2\pi \quad \text{by Lemma 7.1(iii)} \\
&= \left\{2\pi - \sum_{k=1}^{n} \alpha_k\right\} + \left\{\sum_{k=1}^{n} \varepsilon_k - 2\pi\varepsilon(v, G)\right\} \\
&= 2\pi - \sum_{k=1}^{n} \alpha_k \\
&= d_v,
\end{aligned}$$

where the last line follows from the previous one because $\Sigma_{k=1}^{n} \varepsilon_k = 2\pi\varepsilon(v, G)$, by the definition of $\varepsilon(v, G)$. □

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