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# Mod 2 degree and a generalized No Retraction Theorem

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We provide elementary proofs of generalized versions of the No Retraction Theorem and Sperner's Lemma, and a simple definition of mod 2 degree of certain maps.



## 1 Introduction

Three well-known equivalent theorems are the Brouwer Fixed Point Theorem (BFPT), the No Retraction Theorem (NRT) and Sperner's Lemma (SL). The equivalence of the first two is a standard result in topology. The BFPT can be derived directly from SL, which is often proved combinatorially; it is also known that the BFPT implies SL. See [16] for discussion and references of the interrelations of these theorems.

These three theorems have been studied extensively. SL has been generalized in a variety of ways; see, among many references, [1], [3] and [4]. Some authors have related these combinatorial ideas to the notion of topological degree; see [11], [12] and [14]. For a thorough discussion of generalizations of the BFPT, see [2].

The NRT has many well-known proofs, the most common being via algebraic topology. A well-known elementary proof of the NRT, not using algebraic topology, is [8]; a variant on this proof is found in [9]. Another well-known proof of the NRT is given in [6]. This proof is very short and clear, and also avoids algebraic topology, though it uses the Simplicial Approximation Theorem, which makes it not quite as simple as it at first appears. Moreover, it has recently been pointed out in [7] that the standard Simplicial Approximation Theorem does not suffice for [6], and that the Relative Simplicial Approximation Theorem of [17] is needed, thereby diminishing somewhat the elementary nature of this proof of the NRT. Although there are many proofs of the NRT, it is not as widely known that the NRT can be generalized quite simply to a broader class of topological spaces than just balls.

In this note we bring together a number of these ideas, and give simple proofs of two generalizations of the NRT, as well as a generalization of SL. In order to state our results, we need the following standard definition.

**Definition 1.1** Let $K$ be a finite $n$-dimensional simplicial complex. The *boundary* of $K$, denoted $\operatorname{Bd} K$, is the collection of all $(n-1)$-simplices of $K$ that are contained in an odd number of $n$-simplices, together with all the faces of these $(n-1)$-simplices.

Observe that $\operatorname{Bd} K$ is a (possibly empty) $(n-1)$-dimensional subcomplex of $K$. It can be verified that the boundary of a simplicial complex is instrinsic to the underlying space of the simplicial complex (we omit the details), and we can therefore make the following definition.

**Definition 1.2** Let $X$ be a topological space that is homeomorphic to the underlying space of a finite simplicial complex. Let $K$ be a finite simplicial complex, and let $h\colon |K| \to X$ be a homeomorphism. The *boundary* of $X$, denoted $\partial X$, is defined by $\partial X = h(|\operatorname{Bd} K|)$.

Our first generalization of the NRT is as follows.

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**Theorem 1.3** *Let $X$ be a topological space that is homeomorphic to the underlying space of a finite simplicial complex. Then there is no continuous map $r\colon X \to \partial X$ such that $r(x) = x$ for all $x \in \partial X$.*

The only other place that the author has found this version of the generalized NRT is in [13]; the proof in that paper is short and clever, and it avoids both algebraic topology and the Simplicial Approximation Theorem, though it is not entirely transparent (relying on induction over the dimension of the spaces involved). Two slightly less general version of Theorem 1.3 are found in the following references: In [10] there is a proof of the generalized NRT restricted to simplicial complexes that have boundary a sphere, with a different proof than in [13], but one that has the same merits and drawbacks; in [15, pp. 150–151] there is the outline of a proof (given as exercises) of the generalized NRT restricted to pseudomanifolds with boundary, though here the proof uses the Simplicial Approximation Theorem. Our proof of Theorem 1.3 is, we believe, even more elementary and transparent than the proofs given in [10] and [13], avoiding their uses of induction.

Our second generalization of the NRT, given in Theorem 1.4, is not as straightforwardly analogous to the classical NRT as our first generalization, but it does indeed generalize the classical NRT, and we can prove it with very little extra effort (though this proof uses the Simplicial Approximation Theorem). In this case we have slightly less general spaces, but more general maps, than in Theorem 1.3. Theorem 1.4 is not found in [10], [13] or [15, pp. 150–151].

**Theorem 1.4** *Let $X$, respectively $Y$, be a topological space that is homeomorphic to the underlying space of a finite $n$-dimensional, respectively $(n-1)$-dimensional, simplicial complex. Suppose that $\partial X$ and $Y$ are $(n-1)$-manifolds, with $Y$ connected. Let $W$ be a component of $X$. Then there is no continuous map $f\colon X \to Y$ such that $\deg f|\partial W \equiv 1 \pmod 2$.*

Our generalization of Sperner's Lemma is as follows. If $f\colon |K| \to |L|$ is a simplicial map, and if $\sigma$ is a simplex of $L$, then we let $\nu_n(f, \sigma)$ denote the number of $n$-simplices in $K$ that are mapped onto $\sigma$ by $f$.

**Theorem 1.5** *Let $K$ be a finite $n$-dimensional simplicial complex, let $P$ be a finite $n$-pseudomanifold with boundary, and let $f\colon |K| \to |P|$ be a simplicial map such that $f(|\operatorname{Bd} K|) \subseteq |\operatorname{Bd} P|$. Let $\sigma$ be an $(n-1)$-simplex of $\operatorname{Bd} P$, and let $\tau$ be an $n$-simplex of $P$. Then $\nu_n(f, \tau) \equiv \nu_{n-1}(f||\operatorname{Bd} K|, \sigma) \pmod 2$.*

The tools in our proofs of Theorems 1.3 and 1.5 involve nothing more technical than simplicial complexes, subdivision and simplicial maps. The NRT will be seen to be an issue of parity. The key to our proofs is the following very simple result from graph theory, the proof of which is left to the reader (see most any introductory text on graph theory for further details, for example [5, p. 14]). This lemma could almost be said to be the essence of the NRT.

**Lemma 1.6** *Let $G$ be a graph with no loops. The number of vertices of $G$ that are contained in odd numbers of edges is even.*

All simplicial complexes considered will be finite, and will be in Euclidean space; we will not need to specify the dimension of the ambient space. We take all simplices to be closed. If $K$ is a simplicial complex, we let $|K|$ denote the underlying topological space of $K$. If $S$ is a finite set, we will let $|S|$ denote the cardinality of $S$.

## 2 The main lemma

Our main tool is Lemma 2.3 below, which is used in the proofs of all three of our theorems. We start with the following definitions, the first of which gives a broader class of maps than simplicial maps, in that simplices are not required to be mapped onto simplices.

**Definition 2.1** Let $K$ and $L$ be simplicial complexes. A map $f\colon |K| \to |L|$ is *simplex respecting*, abbreviated SR, if the restriction of $f$ to each simplex $\sigma$ of $K$ is an affine linear map taking $\sigma$ into a simplex of $L$.

Note that the composition of two SR maps is SR. Let $f\colon |K| \to |L|$ be an SR map, where $K$ is an $n$-dimensional simplicial complex, and suppose that $\tau$ is an $n$-simplex of $K$. If $f|\tau$ is not injective, and if $x \in f(\tau) - f\big(\tau^{(n-2)}\big)$, then $(f|\tau)^{-1}(x)$ will be a line segment.

**Definition 2.2** Let $K$ and $L$ be finite $n$-dimensional simplicial complexes, and let $f\colon |K| \to |L|$ be an SR map such that $f(|\operatorname{Bd} K|) \subseteq |\operatorname{Bd} L|$. Let $A \subset |L|$ be a polygonal arc with endpoints $x_1$ and $x_2$. We say that $A$ is *proper* with respect to $K$, $L$ and $f$ if the following four conditions hold.

(1) $A \cap f\big(\big|K^{(n-2)}\big|\big) = \emptyset$.

(2) $(A - \{x_1, x_2\}) \cap f\big(\big|K^{(n-1)}\big| - \big|K^{(n-2)}\big|\big)$ consists of a finite set of points, each of which is a vertex of $A$, and at each of which $A$ passes transversally through $f\big(\big|K^{(n-1)}\big| - \big|K^{(n-2)}\big|\big)$.

(3) Each $x_i$ is either in the relative interior of an $n$-simplex of $L$ and not in $f\big(\big|K^{(n-1)}\big|\big)$, or in the relative interior of an $(n-1)$-simplex of $\operatorname{Bd} L$ and not in $f\big(\big|K^{(n-2)}\big|\big)$.

(4) $A \cap |\operatorname{Bd} L|$ contains at most $x_1$ and $x_2$.

For each $x_i$, let the *index* of $f$ at $x_i$, denoted $I_f(x_i)$, be defined as follows: if $x_i$ in the relative interior of an $n$-simplex of $L$, let $I_f(x_i) = \big|f^{-1}(x_i)\big|$; if $x_i$ is in the relative interior of an $(n-1)$-simplex of $\operatorname{Bd} L$, let $I_f(x_i) = \big|(f|\,|\operatorname{Bd} K|)^{-1}(x_i)\big|$.

**Lemma 2.3** *Let $K$ be a finite $n$-dimensional simplicial complex, let $P$ be an $n$-pseudomanifold with boundary, let $f\colon |K| \to |P|$ be an SR map such that $f(|\operatorname{Bd} K|) \subseteq |\operatorname{Bd} P|$, and let $A \subset |P|$ be a polygonal arc with endpoints $x_1$ and $x_2$ that is proper with respect to $K$, $P$ and $f$. Then $I_f(x_1) \equiv I_f(x_2) \pmod 2$.*

Proof. The hypotheses on $A$, and the fact that $f$ is an SR map, together imply that $f^{-1}(A)$ can be thought of as a graph $G$. The vertices of $G$ are all in the inverse images of the vertices of $A$, and the edges of $G$ are of two types, those equal to inverse images of edges of $A$, and those in the inverse images of vertices of $A$. Let $v$ be a vertex of $G$. Thus $v \in f^{-1}(q)$ for some unique vertex $q$ of $A$.

Suppose first that $q \neq x_i$ for $i = 1, 2$. Thus $q \in |P| - |\operatorname{Bd} P|$. If $q \notin f\big(\big|K^{(n-1)}\big| - \big|K^{(n-2)}\big|\big)$, then $v$ is contained in the relative interior of an $n$-simplex of $K$ on which $f$ is injective. Hence $v$ is contained in two edges of $G$, since $q$ is contained in two edges of $A$. If $q \in f\big(\big|K^{(n-1)}\big| - \big|K^{(n-2)}\big|\big)$, then $v$ is in the relative interior of an $(n-1)$-simplex $\eta$ of $K - \operatorname{Bd} K$. If $\tau$ is an $n$-simplex of $K$ containing $\eta$, then either $f|\tau$ is injective, in which case $f(\tau)$ contains at least part of an edge of $A$ (and thus $\tau$ contains at least part of an edge of $G$), or $f|\tau$ is not injective, in which case $\tau$ contains an edge of $G$. It follows that $v$ is contained in one edge of $G$ for each $n$-simplex of $K$ containing $\eta$; since $\eta$ is contained in an even number of $n$-simplices of $K$, then $v$ is contained in an even number of edges of $G$.

Now suppose that $q = x_i$ for some $i$. There are two cases. Suppose first that $q$ is in the relative interior of an $n$-simplex of $P$ and not in $f\big(\big|K^{(n-1)}\big|\big)$. Then $v$ is contained in a single edge of $G$. Second, suppose that $q$ is in the relative interior of an $(n-1)$-simplex of $\operatorname{Bd} P$ and not in $f\big(\big|K^{(n-2)}\big|\big)$. Then $v$ is in the relative interior of an $(n-1)$-simplex $\sigma$ of $K$. If $\sigma$ is in $K - \operatorname{Bd} K$, then as above, it is seen that $v$ is contained in an even number of edges of $G$. On the other hand, if $\sigma$ is in $\operatorname{Bd} K$, then it can be seen that $v$ is contained in an odd number of edges of $G$.

Putting together the above observations, it follows that the number of vertices of $G$ contained in odd numbers of edges of $G$ is equal to $I_f(x_1) + I_f(x_2)$. However, by Lemma 1.6, we deduce that $I_f(x_1) + I_f(x_2)$ is an even number. The desired result follows. □

Our first application of Lemma 2.3 is to prove the following two lemmas, the first of which allows us to give an elementary definition of the mod 2 degree of SR maps, which in turn is used to prove Theorem 1.4.

**Lemma 2.4** *Let $M$ be an $n$-dimensional simplicial complex with $\operatorname{Bd} M = \emptyset$, let $P$ be an $n$-pseudomanifold, and let $f\colon |M| \to |P|$ be an SR map. Let $x_1$ and $x_2$ be points in the relative interiors of $n$-simplices of $P$, such that $x_1, x_2 \notin f\big(\big|M^{(n-1)}\big|\big)$. Then $\big|f^{-1}(x_1)\big| \equiv \big|f^{-1}(x_2)\big| \pmod 2$.*

Proof. By hypothesis we have $|\operatorname{Bd} M| = \emptyset = |\operatorname{Bd} P|$, and hence $f(|\operatorname{Bd} M|) \subseteq |\operatorname{Bd} P|$ is true trivially.

It follows from the strong connectivity in the definition of pseudomanifolds that there is a polygonal arc $A \subset |P|$ with endpoints $x_1$ and $x_2$ satisfying the hypotheses of Lemma 2.3. Hence $I_f(x_1) \equiv I_f(x_2) \pmod 2$. By hypothesis on the $x_i$, we see that $I_f(x_i) = \big|f^{-1}(x_i)\big|$ for $i = 1, 2$. □

**Definition 2.5** Let $M$ be an $n$-dimensional simplicial complex with $\operatorname{Bd} M = \emptyset$, let $P$ be an $n$-pseudomanifold, and let $f\colon |M| \to |P|$ be an SR map. The mod 2 *degree* of $f$, denoted $\deg_2 f$, is the equivalence class in $\mathbb{Z}_2$ of $\big|f^{-1}(x)\big|$ for any $x \in |P|$ in the relative interior of an $n$-simplex of $P$, such that $x \notin f\big(M^{(n-1)}\big)$.

**Lemma 2.6** *Let $K$ be a finite $n$-dimensional simplicial complex, let $N$ be an $(n-1)$-pseudomanifold. Let $C$ be a component of $K$. Then there is no SR map $f\colon |K| \to |N|$ such that $\deg_2 f|\,|\operatorname{Bd} C| = 1$.*

 

Proof. Let $f\colon |K| \to | N|$ be an SR map. We will show that $\deg_2 f||\operatorname{Bd} C| = 0$.

Let $P$ be a cone on $N$. It can be verified that $P$ is an $n$-pseudomanifold with boundary, and that $\operatorname{Bd} P = N$. We can think of $f||C|$ as an SR map $F\colon |C| \to | P|$. Let $x_1$ be a point in the relative interior of an $(n-1)$-simplex of $\operatorname{Bd} P$ and not in $f\big(\big|K^{(n-2)}\big|\big)$, and let $x_2$ be a point in the relative interior of an $n$-simplex of $P$. Let $A$ be the line segment with end points $x_1$ and $x_2$. Then $F$ and $A$ satisfy the hypotheses of Lemma 2.3. Hence $I_F(x_1) \equiv I_F(x_2) \pmod 2$. Clearly $I_F(x_2) = \big|F^{-1}(x_2)\big| = 0$. Therefore $0 = I_F(x_1) = \big|(F||\operatorname{Bd} C|)^{-1}(x_1)\big| = \big|(f||\operatorname{Bd} C|)^{-1}(x_1)\big| = \deg_2 f||\operatorname{Bd} C|$. □

## 3 Proofs of the theorems

Proof of Theorem 1.3. Suppose that $X$ is homeomorphic to $|K|$, for some finite $n$-dimensional simplicial complex $K$. By a remark made above, it will suffice to show that there is no continuous map $r\colon |K| \to |\operatorname{Bd} K|$ such that $r(x) = x$ for all $x \in |\operatorname{Bd} K|$.

If $\operatorname{Bd} K = \emptyset$, then there is no map of any sort $|K| \to | \operatorname{Bd} K|$, so assume $\operatorname{Bd} K \neq \emptyset$. Suppose that there is a continuous map $r\colon |K| \to | \operatorname{Bd} K|$ such that such that $r(x) = x$ for all $x \in |\operatorname{Bd} K|$.

Let $\sigma$ be an $(n-1)$-simplex in $\operatorname{Bd} K$. Let $\Delta$ be an $n$-simplex, let $\eta$ be an $(n-1)$-face of $\Delta$, and let $w$ be the vertex of $\Delta$ not in $\eta$. Define a map $h\colon |K| \to \Delta$ by mapping $\sigma$ affine linearly onto $\eta$, by mapping every vertex in $|K| - \sigma$ to $w$, and extending affine linearly over the simplices of $K$. Note that $h$ is a simplicial map. It is seen that the only $(n-1)$-simplex of $\operatorname{Bd} K$ mapped by $h$ onto $\eta$ is $\sigma$. Let $r' = h \circ r$. Evidently $r'(|K|) \subseteq |\partial\Delta|$, and $(r'||\operatorname{Bd} K|)^{-1}(\eta) = \sigma$ (this observation uses the fact that $r||\operatorname{Bd} K|$ is the identity map).

Let $x_1$ be the barycenter of $\Delta$, and let $\epsilon$ be the distance from $x_1$ to $|\partial\Delta|$. Since $|K|$ is compact, the map $r'$ is uniformly continuous, and thus there is some number $\delta > 0$ such that if $x, y \in |K|$ are any two points such that $\|x - y\| < \delta$, then $\|r(x) - r(y)\| < \epsilon/2$. Let $K'$ be a subdivision of $K$ such that the diameter of any simplex of the subdivision is less than $\delta$. (For example, let $K'$ be the $m$-th barycentric subdivision for large enough $m$.) Using an observation made at the start of this section, it is seen that $|\operatorname{Bd} K'| = |\operatorname{Bd} K|$.

We define a map $F\colon |K'| \to \Delta$ by letting $F(v) = r'(v)$ for all vertices $v$ of $K'$, and extending $F$ affine linearly over all of $K'$. Observe that $F$ is SR, since it maps every simplex of $K'$ affine linearly into $\Delta$. By the choice of $K'$, we see that $x_1 \notin F(|K'|)$. Thus $I_F(x_1) = 0$. Let $x_2$ be the barycenter of $\eta$. Because $(r'||\operatorname{Bd} K|)^{-1}(\eta) = \sigma$, and because $r'$ maps $\sigma$ homeomorphically onto $\eta$, it follows that $I_F(x_2) = 1$. On the other hand, let $A$ be the line segment from $x_1$ to $x_2$. Then $F$ and $A$ satisfy the hypotheses of Lemma 2.3. By the lemma we deduce that $I_F(x_1) \equiv I_F(x_2) \pmod 2$, a contradiction. □

If one wants to prove the classical No Retraction Theorem directly, rather than Theorem 1.3, a simplified version of Lemma 2.3 can be used, giving a shorter proof than we have given.

Theorem 1.4 can be derived easily from Lemma 2.6, by using the Simplicial Approximation Theorem (which we have avoided so far), and the notion of topological degree; we omit the details.

The following very simple example illustrates the role of even vs. odd degree in Theorem 1.4. Let $X$ be the surface with boundary shown in Figure 1 (i); let $B$ be one of the three boundary components of $X$. Then Theorem 1.4 implies that there is no continuous map $f\colon X \to B$ such that $f$ is a homeomorphism when restricted to each boundary component (since then $\deg f|\partial X$ would be 3). By contrast, let $Y$ be the surface with boundary shown in Figure 1 (ii); let $C$ be one of the four boundary components of $Y$. As the reader can verify, there is a continuous map $h\colon Y \to C$ that is a homeomorphism when restricted to each boundary component.

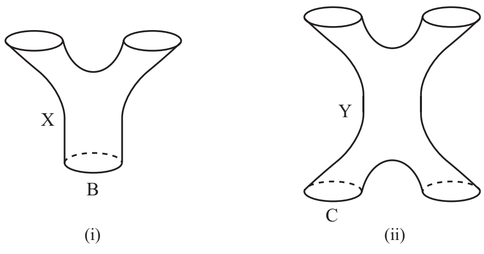


**Fig. 1**

 

Proof of Theorem 1.5. Let $x_1$ be a point in the relative interior of $\sigma$, and let $x_2$ be a point in the relative interior of $\tau$. It follows from the strong connectivity in the definition of pseudomanifolds that there is a polygonal arc $A \subset |P|$ with endpoints $x_1$ and $x_2$ satisfying the hypotheses of Lemma 2.3. Using that lemma, we deduce that $I_f(x_1) \equiv I_f(x_2) \pmod 2$. However, because $f$ is simplicial, we see that $I_f(x_1) = \nu_{n-1}(f|| \operatorname{Bd} K|, \sigma)$ and $I_f(x_2) = \nu_n(f, \tau)$. □

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