# POLYHEDRAL REPRESENTATION OF DISCRETE MORSE FUNCTIONS

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Abstract. It is proved that every discrete Morse function in the sense of Forman on a finite regular CW complex can be represented by a polyhedral Morse function in the sense of Banchoff on an appropriate embedding in Euclidean space of the barycentric subdivision of the CW complex; such a representation preserves critical points. The proof is stated in terms of discrete Morse functions on posets.

## 1. Introduction and Main Idea

In its classical form, Morse theory is concerned with smooth functions on smooth manifolds. See [Mil63] for the basics of Morse theory. In addition to the traditional smooth approach, there are a number of discrete analogs of Morse theory. Two such analogs, both of which have been widely cited in the literature, are due to Banchoff (see [Ban67], [Ban70] and [Ban83]) and, more recently, to Forman (see [For98a], [For98b] and subsequent papers).

Although Banchoff's and Forman's approaches are both widely cited, there does not appear to be in the literature a thorough discussion of the relation between these two approaches. Such a lack of discussion is perhaps due to the the fact that upon first encounter, the two approaches appear to be quite different. Banchoff considers finite polyhedra embedded in Euclidean space, whereas Forman considers CW complexes (not necessarily embedded). A "Morse function" for Banchoff is a projection onto a straight line in Euclidean space, whereas a "Morse function" for Forman is a map, called a "discrete Morse function," that assigns a number to each cell of a CW complex, subject to certain conditions.

Given a projection map, Banchoff defines an index at each vertex of a polyhedron, but at no other cells, whereas given a discrete Morse function,

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Forman defines an index for each critical cell, which could be of any dimension. Banchoff does not define the concept of critical vertices vs. ordinary vertices in [Ban67], though he does do so for polyhedral surfaces in [Ban70], and we will use that approach for all dimensions. For Forman, the distinction between critical cells vs. ordinary cells is of great importance. Finally, Banchoff focuses on relating the index at vertices to polyhedral curvature, whereas Forman focuses on using critical points for the purpose of finding a new CW complex that is homotopy equivalent to the original CW complex, but ideally has fewer cells.

In spite of these apparent differences, the purpose of this note is to prove that for finite regular CW complexes there is a very concrete relation between the approaches of Banchoff and Forman, as given in Theorem 1.3 below. This theorem says that information about critical cells in the sense of Forman can be obtained by Banchoff's method for an appropriate embedding of the barycentric subdivision of the original finite regular CW complex. We note that in Forman's method a critical $p$-cell always has index $p$, and so the only question to be asked is whether or not a cell is critical, not what its index is.

To make our treatment self-contained, we start by recalling the relevant definitions and results of Banchoff and Forman. The reader familiar with [Ban67], the first three sections of [For98a], and the reformulation of Forman's approach in terms of acyclic matchings due to Chari in [Cha00], can skip these definitions and go straight to the statement of Theorem 1.3.

We start our very brief survey with Banchoff's approach. Although he works with embedded convex cell complexes, it suffices for our purposes to work with embedded simplicial complexes.

Let $K$ be a finite simplicial complex in $\mathbb{R}^n$, and let $\xi \in S^{n-1}$ be a unit vector. We assume that $\xi$ is general for $K$, which means that projection of $\mathbb{R}^n$ onto the line spanned by $\xi$ yields distinct values for any two vertices of $K$ that are joined by an edge; this condition is true for almost all values of $\xi$.

Let $v$ be a vertex of $K$. If $\eta^i$ is an $i$-simplex of $K$, let $A(\eta^i, v, \xi) = 1$ if $v \in \eta^i$ and $v$ has the maximal value of all the vertices of $\eta^i$ when projected onto the line spanned by $\xi$, and let $A(\eta^i, v, \xi) = 0$ otherwise.

Banchoff's index at $v$ is defined to be

$$a(v, \xi) = \sum_{\eta^i \in K} (-1)^i A(\eta^i, v, \xi).$$

For our purposes, we need the following equivalent version of this definition. Let $T$ denote the set of all simplices of $K$ that contain $v$ and for which projection onto the line spanned by $\xi$ has maximal value at $v$. The definition of the index can be rewritten as

$$a(v, \xi) = \sum_{\sigma \in T} (-1)^{\dim \sigma}. \tag{1}$$

The following analog of the Gauss-Bonnet Theorem is proved in [Ban67, p. 246].

**Theorem 1.1** (Banchoff)**.** *Let $K$ be a simplicial complex in $\mathbb{R}^n$, let $\xi \in S^{n-1}$ be a unit vector, and let $v$ be a vertex of $K$. If $\xi$ is general for $K$, then*

$$\sum_{v \in K} a(v, \xi) = \chi(K).$$

Banchoff does not define the terms critical vertex vs. ordinary vertex in [Ban67]. In [Ban70], however, which treats only surfaces, he defines a vertex to be critical if and only if $a(v, \xi) \neq 0$, and we will take that definition as the correct one for higher dimensions as well. If we need to emphasize that our use of the term "critical" is in the sense of Banchoff, we will use the term "Banchoff-critical," and similarly for "ordinary."

We now turn to Forman's discrete Morse theory. Froman's approach used CW complexes, rather than just convex cell complexes, and here the Morse functions assign a number to each cell of a CW complex, rather than just to every vertex.

It turns out that the only property of regular CW complexes that is needed for the proof of our main theorem is the fact that the set of cells of a regular CW complex form a graded poset (partially ordered set) in a natural way, and that such a poset has various nice properties. It is therefore more clear, and slightly more general, to discuss Forman's work in the context of posets.

We assume that the reader is familiar with basic properties of posets. See [Sta97, Chapter 3] for details. All posets are assumed to be finite. Let $P$ be a poset. We let $<$ denote the partial order relation on $P$, and we write $a \prec b$ if $b$ covers $a$, and $a \preccurlyeq b$ if $a \prec b$ or $a = b$. If $a \in P$, we let

$$P_{<a} = \{x \in P \mid x < a\} \quad \text{and} \quad P_{\leq a} = \{x \in P \mid x \leq a\}.$$

The order complex of $P$, denoted $\Delta(P)$, is the simplicial complex with a vertex for each element of $P$, and a simplex for each non-empty chain of elements of $P$; it is a standard fact that such a construction yields a simplicial complex. In particular, it is always possible to compute the Euler characteristic $\chi(\Delta(P))$. If $C \subseteq P$ is a chain (always assumed non-empty), we let $l(C)$ denote the length of the chain, which is one less than the number of elements in the chain.

A function $\rho\colon P \to \{0, 1, \ldots, r\}$ for some $r \in \mathbb{N} \cup \{0\}$ is a rank function for $P$ if it satisfies the following conditions: for $a, b \in P$, if $a$ is a minimal element then $\rho(a) = 0$, and if $a \prec b$ then $\rho(a) + 1 = \rho(b)$. A poset is ranked if it has a rank function. (Observe that a ranked poset need not be graded, using the terminology of [Sta97, Chapter 3]; the term "ranked" is used in [RMG$^+$00, Section 11.1.3], though we use the definition of a rank function given in [Sta97, Chapter 3].)

We will also need the following properties of posets, one of which is the mod 2 version of ranked posets, where we partly follow the terminology of [Brä04, p. 6].

**Definition.** Let $P$ be a finite poset.

(1) The poset $P$ is **2-wide** if for any $a, b, c \in P$ such that $a \prec b \prec c$, there is some $d \in P$ such that $d \neq b$ and $a \prec d \prec c$.
(2) Let $\mu \colon P \to \{0, 1\}$ be a function. The function $\mu$ is a **parity rank function** if it satisfies the following conditions: for $a, b \in P$, if $a$ is a minimal element then $\mu(a) = 0$, and if $a \prec b$ then $1 - \mu(a) = \mu(b)$. A poset is **parity-ranked** if it has a parity rank function.
(3) Let $\mu \colon P \to \{0, 1\}$ be a parity rank function. The poset $P$ is **downward Eulerian** if $a \in P$ and $a$ not minimal imply $\chi(\Delta(P_{<a})) = (-1)^{\mu(a)+1} + 1$. △

The poset shown in Figure 1 satisfies these three properties, as the reader may verify.

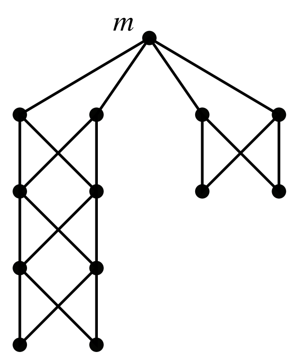



Figure 1

We note that if a finite poset has a rank function, then it is unique, and similarly for a parity rank function.

The downward Eulerian property is weaker than the widely used Eulerian property of posets. As the reader may verify, the poset shown in Figure 2 is downward Eulerian but not Eulerian.

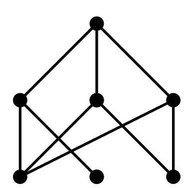


Figure 2

Let $X$ be a regular CW complex. The face poset of $X$, denoted $P(X)$, is the poset that has one element for each cell of $X$, where the order relation is given by $\sigma < \tau$ if $\sigma$ is in the boundary of $\tau$, for cells $\sigma$ and $\tau$ of $X$. The poset $P(X)$ is ranked, where the rank of a cell in $X$ is its dimension. The poset $P(X)$ is 2-wide by [For98a, Theorem 1.2]. The function that assigns each cell of $X$ the number 0 or 1 depending upon whether the dimension of the cell is even or odd is clearly a parity rank function on $P(X)$. The poset $P(X)$ is downward Eulerian, because for each $\sigma \in P(X)$, the interval $P(X)_{<\sigma}$ is the set of all cells in the boundary of $\sigma$, which is a sphere, and hence has the appropriate Euler characteristic. See [LW69] or [Bjö95] for a treatment of CW complexes.

Although the face poset of a regular CW complex is 2-wide, has a parity rank function, and is downward Eulerian, not every poset satisfying these three properties is the face poset of a regular CW complex. For example, whereas the poset shown in Figure 1 satisfies these three properties, this poset is not the face poset of a regular CW complex, because if it were, then the interval $P_{<m}$ would be the face poset of the boundary of cell $m$, and hence $\Delta(P_{<m})$ would be a sphere, and yet $\Delta(P_{<m})$ is not connected.

The following definition is a restatement of Forman's definition in [For98a] in the context of posets.

**Definition.** Let $P$ be a poset, and let $f\colon P \to \mathbb{R}$ be a function.

(1) The map $f$ is a **discrete Morse function** if the following condition holds: for each $b \in P$, there is at most one $a \in P$ such that $a \prec b$ and $f(a) \geq f(b)$, and there is at most one $c \in P$ such that $b \prec c$ and $f(b) \geq f(c)$.

(2) Suppose $f$ is a discrete Morse function. An element $b \in P$ is **critical** with respect to $f$ if there is no $a \in P$ such that $a \prec b$ and $f(a) \geq f(b)$, and there is no $c \in P$ such that $b \prec c$ and $f(b) \geq f(c)$; otherwise $b$ is **ordinary** with respect to $f$. △

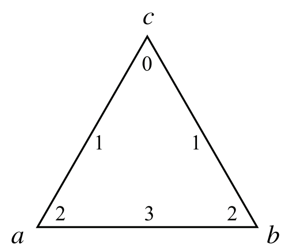
If we need to emphasize that our use of the term "critical" is in the sense of Forman, we will use the term "Forman-critical," and similarly for "ordinary."

The above definition applies to a regular CW complex $X$ by applying it to the face poset of $X$.

An example of a discrete Morse function on the boundary of a triangle is seen in Part (i) of Figure 3, where the number next to each vertex and edge is the value of the function; the critical elements are the vertex $c$ and the edge $ab$. An example that is not a discrete Morse function is seen in Part (ii) of the figure.

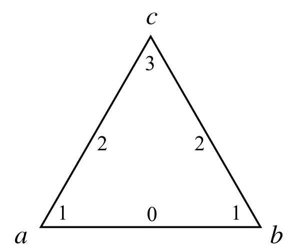
The following lemma is a restatement for posets of Lemma 2.5 of [For98a]; the original proof works for posets.

**Lemma 1.2.** *Let $P$ be a finite poset, and let $f\colon P \to \mathbb{R}$ be a discrete Morse function. Suppose that $P$ is 2-wide. If $b \in P$, there cannot be both some $a \in P$ such that $a \prec b$ and $f(a) \geq f(b)$, and some $c \in P$ such that $b \prec c$ and $f(b) \geq f(c)$.*




Discrete Morse function

(i)




Not a discrete Morse function

(ii)

FIGURE 3

Lemma 1.2 is not true if the assumption that $P$ is 2-wide is dropped. For example, let $P = \{0, 1, 2\}$ have the usual total order, and let $f\colon P \to \mathbb{R}$ be defined by $f(x) = 2 - x$ for $x \in P$. The function $f$ is a discrete Morse function on $P$, but it does not satisfy the conclusion of the lemma.

Whereas the above formulation of discrete Morse functions is Forman's original approach (except that he looked at CW complexes rather than posets), much of the subsequent development and applications of discrete Morse functions has been via a reformulation in terms of acyclic matchings due to Chari in [Cha00]. We now describe this method very briefly, because we will use it; the details may be found in [Cha00] and [Sha01]. For a broader discussion of Chari's aproach as a very useful combinatorial tool, see [Jon08] and [Koz08]. Both these authors dispense with Forman's original formulation, and provide an efficient approach for applications, though it is precisely the fact that both Forman's original approach and Banchoff's approach were inspired by classical Morse theory that led to the question of whether is a relation between the two approaches.

Let $P$ be a poset. We can think of the Hasse diagram of $P$ as a directed graph $G(P)$, where the edges of $G(P)$ are pairs $(a, b) \in P \times P$ such that $a \prec b$, and where an edge of the form $(a, b)$ is directed from $b$ to $a$. A matching $M$ on $G(P)$ is a collection $M$ of edges of $G(P)$ such that if $(a, b) \in M$, then $a$ and $b$ are contained in no other pair in $M$.

Let $f\colon P \to \mathbb{R}$ be a discrete Morse function. Let $b \in P$ be ordinary with respect to $f$. By the definition of discrete Morse functions combined with Lemma 1.2, there is either some $a \in P$ such that $a \prec b$ and $f(a) \geq f(b)$, or there is some $c \in P$ such that $b \prec c$ and $f(b) \geq f(c)$, but not both. We can then match $b$ with either $a$ or $c$, whichever satisfies the above condition. The element $b$ is matched with no other element of $P$, and the element with which $b$ is matched is also ordinary with respect to $f$. Hence, the function $f$ induces a matching on $G(P)$, denoted $M(f)$. The elements of $P$ that are critical with respect to $f$ are precisely those elements of $P$ that are not in any pair in $M(f)$.

The matching $M(f)$ is acyclic, in the following sense. Thinking of $G(P)$ as a directed graph, we form a new directed graph $G_{M(f)}(P)$ by reversing the directions of all the edges in $M(f)$. The graph $G_{M(f)}(P)$ is acyclic in the usual sense for directed graphs. See [Sha01, Section 2] for details. Conversely, for any acyclic matching $N$ on $G(P)$, there exists a discrete Morse function $g\colon P \to \mathbb{R}$ such that $N = M(g)$. This fact is also discussed in [Sha01, Section 2], but a proof can be found inside the proof of Lemma 2.3 below, where we define a specific discrete Morse function that meets our needs.

Our main result is as follows.

**Theorem 1.3.** *Let $X$ be a finite regular CW complex, and let $f$ be a discrete Morse function on $X$. For any sufficiently large $m \in \mathbb{N}$, and for any unit vector $\xi \in S^{m-1}$, there is a polyhedral embedding of the barycentric subdivision of $X$ in $\mathbb{R}^m$ such that a cell in $X$ is Forman-critical with respect to $f$ if and only if its barycenter is Banchoff-critical with respect to projection onto the line spanned by $\xi$.*

We take the barycentric subdivision of the CW complex in Theorem 1.3 for the following reasons. First, whereas Forman's method determines whether each cell is critical or ordinary, Banchoff's method assigns such information only to the vertices, and by taking the barycentric subdivision of a CW complex we obtain a single vertex corresponding to each original cell. Second, the barycentric subdivision of a regular CW complex is a simplicial complex, and simplicial complexes are easier to embed in Euclidean space than more general cell complexes.

Third, even if the original CW complex were a simplicial complex, we would still need to take its barycentric subdivision *prior to* embedding the complex in Euclidean space, because of the following simple example. Let $K$ be a triangle together with its faces, which is a simplicial complex. The function that assigns to each face of the triangle its dimension is a discrete Morse function, and every face is Forman-critical with respect to this discrete Morse function, as mentioned in [For98a, p. 108]. Suppose that $K$ is embedded in Euclidean space prior to barycentric subdivision, and is then barycentrically subdivided. The projection onto any general line in the Euclidean space takes any point in the interior of an edge of $K$ to a value lower than one of its vertices and greater than the other its vertices, and it can therefore be seen that under any such projection, the barycenter of an edge in $K$ is Banchoff-ordinary. Hence, if we want to recover the Forman-critical cells of $K$ by projection onto a line in Euclidean space, we need the flexibility of first taking the barycentric subdivision prior to embedding in Euclidean space.

Finally, it is noted that Theorem 1.3 does not imply that Banchoff's Morse theory is somehow "as strong as" Forman's Morse theory, because the two theories do different things. In [AB12], by contrast, it is stated that "Formans discretization of Morse theory is sometimes sharper than the original theory, in bounding the homology groups of a manifold." However, no such comparison can be made between Banchoff's and Forman's approaches, even with Theorem 1.3, because Banchoffs method does not say anything about bounding the homology of a manifold; Banchoff relates his Morse theory to curvature and the Euler characteristic, not homotopy type as does Forman.

## 2. Proof of the Theorem

We have three lemmas, the first two of which are very simple, and the third of which is the bulk of our work. Theorem 1.3 will be seen to be an immediate corollary of Lemma 2.3, by using the properties of the face poset of a CW complex stated above, and the definition of Banchoff-critical.

**Lemma 2.1.** *Let $V$ be a finite set with $n$ elements, where $n \geq 1$, and let $f\colon V \to \mathbb{R}$ be a function. There is a map $\psi\colon V \to \mathbb{R}^n$ such that $\psi(V)$ spans an $(n-1)$-simplex, and such that for each vertex $v \in V$, the projection of $\psi(v)$ onto the $x$-axis equals $f(v)$.*

*Proof.* The proof is by induction on $n$. If $n = 1$, let $v$ be the single element of $V$, and then define $\psi(v) \in \mathbb{R}$ to be $\psi(v) = f(v)$. Now suppose the result is true for $n-1$, where $n \geq 2$. Let $w \in V$, and let $V' = V - \{w\}$. Because $V'$ has at least one element, then by the inductive hypothesis there is a map $\phi\colon V' \to \mathbb{R}^{n-1}$ such that $\phi(V')$ spans an $(n-2)$-simplex, and that for each vertex $v \in V'$, the projection of $\phi(v)$ onto the $x$-axis equals $f(v)$. We can think of $\mathbb{R}^{n-1}$ as sitting in $\mathbb{R}^n$ in the usual way, and hence we can think of $\phi$ as a map $V' \to \mathbb{R}^n$. Let $\psi\colon V \to \mathbb{R}^n$ be defined by letting $\psi|_{V'} = \phi$, and letting $\psi(w)$ be a point in $\mathbb{R}^n$ with first coordinate equal to $f(w)$, and last coordinate not equal to zero. Because $\psi(w)$ can be joined to $\psi(V')$, we see that $\psi(V)$ spans an $(n-1)$-simplex, and it is evident by definition that for each vertex $v \in V$, the projection of $\psi(v)$ onto the $x$-axis equals $f(v)$. $\square$

For the next lemma, we need the following notation. Let $P$ be a poset. If $S \subseteq P$, we let chains$(S)$ denote the set of non-empty chains in $S$. If $b, s, t \in P$, and if $s \preccurlyeq b$ and $t \preccurlyeq b$, we let

$$\begin{aligned} ch(b;\, s) &= \{C \in \text{chains}(P_{\leq b}) \mid s \in C\} \\ ch(b;\, s,\, t) &= \{C \in \text{chains}(P_{\leq b}) \mid s \in C \text{ and } t \in C\} \\ ch(b;\, \neg s,\, t) &= \{C \in \text{chains}(P_{\leq b}) \mid s \notin C \text{ and } t \in C\}. \end{aligned}$$

**Lemma 2.2.** *Let $P$ be a poset. Suppose that $P$ is 2-wide, is parity-ranked with parity rank function $\mu$, and is downward Eulerian. Let $a, b \in P$.*

(1) $\displaystyle\sum_{C \in ch(b;\, b)} (-1)^{l(C)} = (-1)^{\mu(b)}$.

(2) *If $a \prec b$, then* $\displaystyle\sum_{C \in ch(b;\, \neg a,\, b)} (-1)^{l(C)} = 0$.

(3) *If $a \prec b$, then* $\displaystyle\sum_{C \in ch(b;\, a)} (-1)^{l(C)} = 0$.

*Proof.* For Part (1), there are two cases. First, suppose that $b$ is a minimal element of $P$. Therefore $\mu(b) = 0$. Also, we see that $ch(b;\, b) = \{\{b\}\}$, and therefore $\sum_{C\in ch(b;\,b)}(-1)^{l(C)} = (-1)^0 = (-1)^{\mu(b)}$. Second, suppose $b$ is not a minimal element. There is a bijective map from $ch(b;\, b) - \{\{b\}\}$ to chains$(P_{<b})$, where the map is obtained by taking each chain in the former set and removing $b$. This map shortens the length of each chain by 1. Using the definition of the order complex together with the definition of downward Eulerian, we have

$$\begin{aligned}\sum_{C\in ch(b;\,b)}(-1)^{l(C)} &= \sum_{D\in \mathrm{chains}(P_{<b})}(-1)^{l(D)+1} + (-1)^{l(\{b\})}\\ &= -\sum_{D\in \mathrm{chains}(P_{<b})}(-1)^{l(D)} + (-1)^0 = -\chi(\Delta(P_{<b})) + 1\\ &= -\left[(-1)^{\mu(b)+1} + 1\right] + 1 = (-1)^{\mu(b)}.\end{aligned}$$

For Part (2), we observe that

$$ch(b;\, \neg a,\, b) = ch(b;\, b) - ch(b;\, a,\, b).$$

There is a bijective map from $ch(b;\, a,\, b)$ to $ch(a;\, a)$, where the map is obtained by taking each chain in the former set and removing $b$. This map shortens the length of each chain by 1. We then use Part (1), together with the fact that $\mu(a) = 1 - \mu(b)$, to see that

$$\begin{aligned}\sum_{C\in ch(b;\,\neg a,\,b)}(-1)^{l(C)} &= \sum_{C\in ch(b;\,b)}(-1)^{l(C)} - \sum_{D\in ch(b;\,a,\,b)}(-1)^{l(D)}\\ &= \sum_{C\in ch(b;\,b)}(-1)^{l(C)} - \sum_{D\in ch(a;\,a)}(-1)^{l(D)+1}\\ &= (-1)^{\mu(b)} - \left[-(-1)^{\mu(a)}\right] = 0.\end{aligned}$$

The proof of Part (3) is similar to the proof of Part (2), and we omit the details. ☐

**Lemma 2.3.** *Let $P$ be a finite poset, and let $f\colon P \to \mathbb{R}$ be a discrete Morse function. Suppose that $P$ is 2-wide, is parity-ranked with parity rank function $\mu$, and is downward Eulerian. For any sufficiently large $m \in \mathbb{N}$, and for any unit vector $\xi \in S^{m-1}$, there is a polyhedral embedding $\phi\colon \Delta(P) \to \mathbb{R}^m$ such that the projection of $\mathbb{R}^m$ onto the line spanned by $\xi$ is general for $\phi(\Delta(P))$, and such that for every $b \in P$, the index of $\phi(b)$ with respect to this projection is given by*

$$a(\phi(b), \xi) = \begin{cases}(-1)^{\mu(b)}, & \text{if } b \text{ is Forman-critical with respect to } f\\ 0, & \text{if } b \text{ is Forman-ordinary with respect to } f.\end{cases} \tag{2}$$

*Proof.* We show that an embedding with the desired property can be found for a single choice of $\mathbb{R}^m$ and with respect to $\xi$ being the unit vector in the direction of the positive $x$-axis. It will then follow immediately that an appropriate embedding can be found in $\mathbb{R}^k$ for $k > m$ with respect to the same $\xi$ by using using the usual embedding of $\mathbb{R}^m$ in $\mathbb{R}^k$. Appropriate embeddings with respect to any unit vector $\xi \in S^{k-1}$ can be found by rotating and translating the original embedding.

We start by defining a function $g\colon P \to \mathbb{R}$ as follows. As discussed above, the discrete Morse function $f$ induces an acyclic matching $M(f)$ on $G(P)$. By [Koz08, Theorem 11.2], there is a linear extension $L$ of $P$ such that if $(a, b) \in M(f)$, then $a \prec_L b$, where $\prec_L$ means covered in the order $L$. Let $h\colon P \to \mathbb{R}$ be any function that is strictly increasing with respect to the order $L$. Let $g$ be defined by taking every pair $(a, b) \in M(f)$, and letting $g(a) = h(b)$ and $g(b) = h(a)$, and having $g$ equal $h$ on all elements of $P$ that are not in any pair in $M(f)$. This definition works because each element of $P$ is in at most one pair in $M(f)$.

The function $g$ satisfies the following four properties.

(1) The function $g$ is injective.
(2) If $x, y \in P$, and $x < y$ and $x \not\prec y$, then $g(x) < g(y)$.
(3) The function $g$ is a discrete Morse function.
(4) An element of $P$ is critical with respect to $f$ if and only it is critical with respect to $g$.

Properties (1) and (2) are immediate, because $h$ is strictly increasing, and because of the way $g$ is defined in terms of $h$. Properties (3) and (4) both follow from the correspondence between discrete Morse functions and acyclic matchings, and the fact that both functions correspond to the same acyclic matching on $G(P)$.

By Properties (3) and (4), it will suffice to prove the lemma with $f$ replaced by $g$.

Suppose that $P$ has $k$ elements. By Lemma 2.1 there is a map $\psi\colon P \to \mathbb{R}^k$ such that $\psi(P)$ spans an $(k-1)$-simplex, and that for each vertex $v \in V$, the projection of $\psi(v)$ onto the $x$-axis equals $g(v)$. Because $\Delta(P)$ is a simplicial complex with $k$ vertices, it can be identified with a subcomplex of the $(k-1)$-simplex spanned by $\psi(P)$. Hence we can think of $\psi$ as inducing a polyhedral embedding $\phi\colon \Delta(P) \to \mathbb{R}^k$, where $\phi(v) = \psi(v)$ for all $v \in P$, and where we think of $P$ as the set of vertices of $\Delta(P)$.

Because of Property (1), we see that if $\psi(a)$ and $\psi(b)$ are vertices of $\Delta(P)$ that are joined by an edge, then $g(a) \neq g(b)$, and hence the projection of $\psi(a)$ onto the $x$-axis does not equal the projection of $\psi(b)$ onto the $x$-axis. We can therefore define Banchoff's index at the vertices of $\phi(\Delta(P))$, where the projection is onto the $x$-axis, and hence we can apply the notion of Banchoff-critical and Banchoff-ordinary to these vertices.

Let $b \in P$, so that $b$ is a vertex of $\Delta(P)$. We compute the index $a(\phi(b), \xi)$ as discussed in Section 1. Let $T$ denote the set of all simplices of $\Delta(P)$ that contain $\phi(b)$ as a vertex and for which projection onto the $x$-axis has maximal value at $\phi(b)$. By Equation 1 we have $a(\phi(b), \xi) = \sum_{s \in T} (-1)^{\dim s}$. We can view this last formula from a different perspective. By the definition of $\Delta(P)$, every simplex of $\Delta(P)$ is a non-empty chain in $P$. The choice of $\phi$ states that the projection of $\phi(a)$ onto the $x$-axis equals $g(a)$ for all $a \in P$. Hence, we see we can think of $T$ as the set of all chains in $P$ that contain $b$, and on which $g$ is maximal at $b$. If $C$ is a chain in $P$, then the dimension of this chain when thought of as a simplex of $\Delta(P)$ is equal to $l(C)$. Therefore $a(\phi(b), \xi) = \sum_{C \in T} (-1)^{l(C)}$.

Suppose that $b$ is critical with respect to $g$. Let $v \in P$ be such that $v < b$. If $v \prec b$, then $g(v) < g(b)$ because $b$ is critical with respect to $g$. If $v \not\prec b$, then $g(v) < g(b)$ by Property (2). A similar argument shows that $g(b) < g(u)$ for any $u \in P$ such that $b < u$. Hence, the set $T$ consists precisely of all chains in $P_{\leq b}$ that contain $b$; this set is denoted $ch(b;\, b)$. Lemma 2.2 (1) implies that

$$a(\phi(b), \xi) = \sum_{C \in T} (-1)^{l(C)} = \sum_{C \in ch(b;\, b)} (-1)^{l(C)} = (-1)^{\mu(b)}.$$

Next, suppose that $b$ is ordinary with respect to $g$. By Lemma 1.2 either there is a single $h \in P$ such that $h \prec b$ and $g(h) \geq g(b)$, or there is a single $u \in P$ such that $b \prec u$ and $g(b) \geq f(u)$, but not both.

First, suppose that there is some $h \in P$ such that $h \prec b$ and $g(h) \geq g(b)$. It follows that $g(b) < g(z)$ for all $z \in P$ such that $b \prec z$. By the same argument used above, we know that $g(b) < g(u)$ for any $u \in P$ such that $b < u$, and hence that $T \subseteq ch(b;\, b)$.

Let $c \in P$ be such that $c < b$ and $c \neq h$. If $c \prec b$, then the definition of discrete Morse functions implies that $g(c) < g(b)$. If $c \not\prec b$, then $g(c) < g(b)$ by Property (2).

Putting the above considerations together, we see that $h$ is the only element of $P_{\leq b}$ such that $g(h) > g(b)$. Hence $T = ch(b;\, \neg h,\, b)$, and Lemma 2.2 (2) implies that

$$a(\phi(b), \xi) = \sum_{C \in ch(b;\, \neg h,\, b)} (-1)^{l(C)} = 0.$$

Second, suppose that there is some $u \in P$ such that $b \prec u$ and $g(b) \geq g(u)$. An argument similar to the previous case shows that $T = ch(u;\, b)$. Lemma 2.2 (3) implies that $a(\phi(b), \xi) = 0$. $\square$

We note that it is possible to prove Lemma 2.3, and hence Theorem 1.3, using only Forman's original approach, but without using acyclic matchings and [Koz08, Theorem 11.2], which were used to define the function $g$ in the above proof. However, the additional length of doing so outweighs the saving in length by virtue of not needing the proof that discrete Morse functions correspond to acyclic matching and the proof of [Koz08, Theorem 11.2].

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