# Functions of Finite Simplicial Complexes That Are Not Locally Determined

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**Abstract**

The Euler characteristic, thought of as a function that assigns a numerical value to every finite simplicial complex, is locally determined in both a combinatorial sense and a geometric sense. In this note we show that not every function that assigns a numerical value to every finite simplicial complex via a linear combination of the numbers of simplices in each dimension is locally determined in either sense. In particular, the Charney-Davis quantity $\lambda(L)$ is not locally determined in either sense if it is defined on a set of simplicial complexes that includes all flag spheres of a given odd dimension.

## 1 Introduction

There are a number of contexts in which to consider the notion that a function that assigns a numerical value to every finite simplicial complex (for example the Euler characteristic) is locally determined. We discuss two approaches here. The common idea to both approaches is that if $\mathcal{T}$ is a set of finite simplicial complexes, and if $\Lambda$ is a real-valued function on $\mathcal{T}$, the function $\Lambda$ is locally determined if there is an appropriate type of real-valued function $c$ defined at each vertex of each simplicial complex in $\mathcal{T}$ such that

$$\Lambda(K) = \sum_{v \in K^{(0)}} c(v)$$

for every $K \in \mathcal{T}$, where $K^{(0)}$ is the set of vertices of $K$.

In the combinatorial approach of [14] and [8], described below in Section 2, the number $c(v)$ depends only upon the combinatorial nature of the link of $v$ in $K$. For example, it is observed in those two papers that the Euler characteristic (ordinary, not reduced) on the set of all simplicial complexes is combinatorially locally determined by the real valued function with formula $c(v) = e(\mathrm{link}\,(v, K))$, where the function $e$ on the set of all simplicial complexes is given by

$$e(M) = 1 + \sum_i \frac{(-1)^{i+1}}{i+2} f_i(M)$$

for all simplicial complexes $M$.

There is also a geometric notion of a function on a set of finite simplicial complexes being locally determined, as discussed in [4], and described below in Section 3. This approach is inspired by the polyhedral analog of the Gauss-Bonnet Theorem, which is as follows. Let $M$ be a finite polyhedral surface in $\mathbb{R}^3$. If $v \in M^{(0)}$, the angle defect at $v$ is defined to be $d_v = 2\pi - \sum \alpha_i$, where the $\alpha_i$ are the angles of the polygons containing $v$. The polyhedral analog of the Gauss-Bonnet Theorem says $\sum_{v \in M^{(0)}} d_v = 2\pi\chi(M)$. Rather than viewing this formula as stating that the angle defects at the vertices add up to something nice, it can be viewed as stating that the Euler characteristic of the surface is locally determined by a geometrically calculated quantity. The polyhedral Gauss-Bonnet Theorem can be generalized to higher dimensions and to non-manifolds in more than one way, as seen, among others, in [2], [6], [16], [10], [11] and [3]; we take the approach of the latter. For Descartes' original work on the angle defect see [7]; for a very accessible treatment of the angle defect see [1].

In contrast to the combinatorial approach of [14] and [8], where the local calculation at each vertex of a simplicial complex depends only upon the combinatorial nature of the link (or star) of the vertex, in the geometric approach the local calculation at a vertex depends upon local geometric information that makes use of an embedding of the simplicial complex in Euclidean space. Hence, in the geometric approach, rather than considering simplicial complexes to be the same if they are combinatorially equivalent (as in [14] and [8]), we consider simplicial complexes that are embedded in Euclidean space, and view different embeddings of the same abstract simplicial complex as different.

The Euler characteristic is not only locally determined in both the combinatorial and geometric senses, but it is the unique function that is locally determined and that satisfies some additional conditions. In the combinatorial approach, [14] and [8] show that the Euler characteristic is, up to a scalar multiple, the unique combinatorially locally determined numerical invariant of finite simplicial complexes that assigns the same number to every cone; that would hold, in particular, for a numerical invariant that is a homotopy type invariant. In the geometric approach, [4] shows in the 2-dimensional case that the Euler characteristic is, up to a scalar multiple, the unique geometrically locally determined numerical invariant of finite simplicial surfaces that assigns the same number to every pyramid and bipyramid.

However, whereas the Euler characteristic is a very useful example, there are combinatorially invariant ways to assign a numerical value to every finite simplicial complex that are not constant on all cones (not to mention are not homotopy type invariants). An example of such a function is the Charney-Davis quantity $\lambda(K)$, as defined in [5].

To define the Charney-Davis quantity and state our result about it, we start with some notation and a definition. (For basic definitions regarding simplicial complexes, see for example [13] and [15].) Throughout this note, all simplicial complexes are assumed to be finite. Let $K$ be a finite simplicial complex. Let $|K|$ denote the underlying space of $K$, and let $K^{(0)}$ denote the set of vertices of $K$. Let $f_i(K)$ denote the number of $i$-simplices of $K$ for each $i \in \{0, 1, \ldots, \dim K\}$; we also use the standard convention that $f_{-1}(K) = 1$, and $f_i(K) = 0$ for each $i \in \mathbb{Z} - \{-1, 0, 1, \ldots, \dim K\}$. If $v \in K^{(0)}$, let $\mathrm{star}\,(v, K)$ and $\mathrm{link}\,(v, K)$ denote the star and link of $v$ in $K$, respectively.

**Definition.** Let $K$ be a simplicial complex. The simplicial complex $K$ is a **flag complex** if for any subset $T \subseteq K^{(0)}$, if every two distinct vertices of $T$ are joined by an edge then $T$ is the set of vertices of a face of $K$. $\triangle$

The Charney-Davis quantity $\lambda(K)$ is defined to be

$$\lambda(K) = \sum_{i=-1}^{\dim K} \left(-\tfrac{1}{2}\right)^{i+1} f_i(K).$$

If a function such as the Charney-Davis quantity were locally defined in either the combinatorial or the geometric approach, that might provide a useful tool for its study. Unfortunately, as seen in Corollaries 2.2 and 3.2, the Charney-Davis quantity defined on odd-dimensional simplicial flag spheres is not locally determined in either sense. The requirement of odd-dimensional simplicial flag spheres here is appropriate, because the Charney-Davis Conjecture, stated in Conjecture D of [5], concerns the value of the Charney-Davis quantity on odd-dimensional simplicial flag generalized homology spheres.

The Charney-Davis quantity is an example of a real-valued function $\Lambda$ on a set of finite simplicial complexes $\mathcal{T}$ that has the form $\Lambda(K) = \sum_{i=-1}^{\dim K} b_i f_i(K)$ for all $K \in \mathcal{T}$, for some $b_{-1}, b_0, b_1, \ldots \in \mathbb{R}$. We ask which such functions $\Lambda$ are locally determined.

Our main result, stated in the combinatorial case in Theorem 2.1 and in the geometric case in Theorem 3.1, gives some criteria under which functions are locally determined in either sense. Our result regarding the Charney-Davis quantity follows immediately from these two theorems.

## 2 Combinatorially Locally Determined Functions

For the combinatorial approach of [14] and [8], we need the following notation. If $\mathcal{T}$ is a set of simplicial complexes, let

$$\mathrm{LI}(\mathcal{T}) = \{\mathrm{link}\,(v, K) \mid K \in \mathcal{T} \text{ and } v \in K^{(0)}\}.$$

The idea of a function on a set of simplicial complexes being locally determined is that the value of the function of a simplicial complex equals the sum of the values of some other function calculated in a "neighborhood" of each vertex of the simplicial complex. The standard notion of neighborhood of a vertex in a simplicial complex is the star of the vertex, but the star of the vertex is determined by the link of the vertex, and in the following definition, which is from [14] and [8], it is convenient to use the link rather than the star.

**Definition.** Let $\mathcal{T}$ be a set of simplicial complexes, and let $\Lambda$ be a real-valued function on $\mathcal{T}$. The function $\Lambda$ is **combinatorially locally determined** by a real-valued function $\delta$ on $\mathrm{LI}(\mathcal{T})$ if $\delta$ is invariant under combinatorial equivalence and if

$$\Lambda(K) = \sum_{v \in K^{(0)}} \delta(\mathrm{link}\,(v, K))$$

for every $K \in \mathcal{T}$. $\triangle$

The adverb "combinatorially" in the above definition is not used in [14] and [8], but for the sake of clarity it seems appropriate to use it at present.

Our result in the combinatorial setting is the following.

**Theorem 2.1.** *Let $\mathcal{T}$ be a set of finite simplicial complexes, and let $\Lambda$ be a real-valued function on $\mathcal{T}$. Suppose that $\Lambda$ has the form $\Lambda(K) = \sum_{i=-1}^{\dim K} b_i f_i(K)$ for all $K \in \mathcal{T}$, for some $b_{-1}, b_0, b_1, \ldots \in \mathbb{R}$.*

1. *If all the simplicial complexes in $\mathcal{T}$ have the same non-zero Euler characteristic, then $\Lambda$ is combinatorially locally determined.*

2. *If $b_{-1} = 0$, then $\Lambda$ is combinatorially locally determined.*

3. *If $b_{-1} \neq 0$, and if $\mathcal{T}$ contains all flag $d$-spheres for some odd integer $d$ such that $d \geq 3$, then $\Lambda$ is not combinatorially locally determined.*

*Proof.* Parts (1) and (2) are very simple. For each case, we will define a sequence $a_{-1}, a_0, a_1, \ldots \in \mathbb{R}$, and we will define a real-valued function $\delta$ on $\mathrm{LI}(\mathcal{T})$ that has the form $\delta(M) = \sum_{i=-1}^{\dim M} a_i f_i(M)$ for each $M \in \mathrm{LI}(\mathcal{T})$.

First, we make the following observation. Let $K \in \mathcal{T}$, let $v \in K^{(0)}$ and let $i \in \{-1, \ldots, \dim K - 1\}$. It is straightforward to see that $\sum_{v \in K^{(0)}} f_i(\mathrm{link}\,(v, K)) = (i+2) f_{i+1}(K)$. Then

$$\begin{aligned} \sum_{v \in K^{(0)}} \delta(\mathrm{link}\,(v, K)) &= \sum_{v \in K^{(0)}} \sum_{i=-1}^{\dim K - 1} a_i f_i(\mathrm{link}\,(v, K)) \\ &= \sum_{i=-1}^{\dim K - 1} a_i \sum_{v \in K^{(0)}} f_i(\mathrm{link}\,(v, K)) \qquad (1) \\ &= \sum_{i=-1}^{\dim K - 1} a_i (i+2) f_{i+1}(K). \end{aligned}$$

For Part (1), where we assume that all the simplicial complexes in $\mathcal{T}$ have Euler characteristic $E$, for some $E \neq 0$, let $a_i = \frac{b_{i+1}}{i+2} + (-1)^{i+1} \frac{b_{-1}}{E(i+2)}$ for all $i \in \{-1, 0, 1, \ldots\}$. For Part (2), where we assume that that $b_{-1} = 0$, let $a_i = \frac{b_{i+1}}{i+2}$ for all $i \in \{-1, 0, 1, \ldots\}$. In both cases, it is straightforward to verify that Equation (1) implies $\sum_{v \in K^{(0)}} \delta(\mathrm{link}\,(v, K)) = \sum_{i=-1}^{\dim K} b_i f_i(K) = \Lambda(K)$; the details are omitted.

For Part (3), suppose that $b_{-1} \neq 0$, and that there is some odd integer $d$ such that $d \geq 3$ and that $\mathcal{T}$ contains all flag $d$-spheres. Suppose further that $\Lambda$ is locally determined by a real-valued function $\delta$ on $\mathrm{LI}(\mathcal{T})$.

We need the following basic facts about joins of simplicial complexes. Let $K$ and $L$ be finite simplicial complexes, and let $K * L$ denote the join of $K$ and $L$. Then $\dim(K * L) = \dim K + \dim L + 1$, and

$$f_r(K * L) = \sum_{i=-1}^{r} f_{r-i-1}(K) f_i(L). \qquad (2)$$

for each $r \in \mathbb{Z}$. Additionally, if $v \in K^{(0)}$, then

$$\operatorname{link}(v, K * L) = \operatorname{link}(v, K) * L. \tag{3}$$

Let $n, m \in \mathbb{N}$ be such that $n, m \geq 4$ and $n \neq m$. Let $C_n$ denote the cycle with $n$ vertices (that is, the graph).

Let $s, t \in \mathbb{N} \cup \{0\}$. Let

$$T_{s,t} = \underbrace{C_n * \cdots * C_n}_{s \text{ times}} * \underbrace{C_m * \cdots * C_m}_{t \text{ times}}.$$

Then $T_{s,t}$ is a $(2(s+t)-1)$-dimensional simplicial complex. We note that $T_{s,t}$ is a flag complex because $C_n$ and $C_m$ are flag complexes (because $n, m \geq 4$), and the join of flag complexes is a flag complex (Item 2.7.1 of [5]). We also see that $T_{s,t}$ is a sphere, because the join of spheres is a sphere (Lemma 1.13 of [13]).

It can be verified that

$$f_i(T_{s,0}) = \sum_{j=0}^{s} \tbinom{s}{j} \tbinom{j}{2j-i-1} n^j$$

for all $i \in \mathbb{Z}$; if $i \in \{-1, 0, \ldots, 2s-1\}$, count the ways $i$-dimensional simplices in this join can be formed, and if $i \notin \{-1, 0, \ldots, 2s-1\}$, then the formula correctly yields $f_i(T_{s,0}) = 0$. A similar formula holds for $f_i(T_{0,t})$. Then, using Equation (2), we see that

$$\begin{aligned} f_r(T_{s,t}) = f_r(T_{s,0} * T_{0,t}) &= \sum_{i=-1}^{r} f_{r-i-1}(T_{s,0}) f_i(T_{0,t}) \\ &= \sum_{i=-1}^{r} \sum_{j=0}^{s} \sum_{k=0}^{t} \tbinom{s}{j} \tbinom{j}{2j+i-r} \tbinom{t}{k} \tbinom{k}{2k-i-1} n^j m^k \end{aligned} \tag{4}$$

for all $r \in \mathbb{Z}$.

Let $v \in (T_{s,t})^{(0)}$. Then $v$ is either in a copy of $C_n$ or a copy of $C_m$. Let $S^0$ denote a two-element set. If $v$ is in a copy of $C_n$, then by Equation (3) we see that $\operatorname{link}(v, T_{s,t})$ is isomorphic to $S^0 * T_{s-1,t}$, and we let $A_{s-1,t} = \delta(\operatorname{link}(v, T_{s,t})) = \delta(S^0 * T_{s-1,t})$; similarly, if $v$ is in a copy of $C_m$, we let $A_{s,t-1} = \delta(\operatorname{link}(v, T_{s,t})) = \delta(S^0 * T_{s,t-1})$.

Let $p \in \mathbb{N}$ be such that $d = 2p-1$. Then $p \geq 2$. Let $u \in \{0, 1, \ldots, p\}$. Then $T_{p-u,u}$ is a $(2p-1)$-dimensional simplicial flag sphere, and hence

$T_{p-u,u} \in \mathcal{T}$. Using the fact that $\Lambda$ is locally determined by $\delta$, we have

$$\begin{aligned}\sum_{r=-1}^{2p-1} b_r f_r(T_{p-u,u}) = \Lambda(T_{p-u,u}) &= \sum_{v \in (T_{p-u,u})^{(0)}} \delta(\operatorname{link}(v, T_{p-u,u})) \\ &= f_0(T_{p-u,0})A_{p-u-1,u} + f_0(T_{0,u})A_{p-u,u-1}.\end{aligned} \tag{$E_u$}$$

Next, we take Equations $(E_0)$, $(E_1)$, …, $(E_p)$, and form the linear combination $\sum_{w=0}^{p}(-1)^w \binom{p}{w}\left(\frac{m}{n}\right)^{p-w} E_w$, which, after rearranging, yields

$$\begin{aligned}&\sum_{r=-1}^{2p-1} b_r \sum_{w=0}^{p} (-1)^w \binom{p}{w} \left(\tfrac{m}{n}\right)^{p-w} f_r(T_{p-w,w}) \\ &\qquad = \sum_{w=0}^{p} (-1)^w \binom{p}{w} \left(\tfrac{m}{n}\right)^{p-w} \left[f_0(T_{p-w,0})A_{p-w-1,w} + f_0(T_{0,w})A_{p-w,w-1}\right].\end{aligned} \tag{5}$$

We now simplify Equation (5), starting with the right-hand side of the equation, which is a linear combination of terms of the form $A_{p-u,u-1}$, where $u \in \{0, 1, \ldots, p+1\}$. Each such term appears twice, once in each of Equations $(E_{u-1})$ and $(E_u)$. (It might be thought that each of $A_{p,-1}$ and $A_{-1,p}$ appear only once, in Equations $(E_0)$ and $(E_p)$, respectively, but each of $A_{p,-1}$ and $A_{-1,p}$ has coefficient $f_0(T_{0,0})$, which is zero, and so we can ignore these terms.) The sum of the two coefficients of $A_{p-u,u-1}$ from Equations $(E_{u-1})$ and $(E_u)$ is

$$\begin{aligned}&(-1)^{u-1}\binom{p}{u-1}\left(\tfrac{m}{n}\right)^{p-(u-1)} f_0(T_{p-(u-1),0}) + (-1)^u \binom{p}{u}\left(\tfrac{m}{n}\right)^{p-u} f_0(T_{0,u}) \\ &\qquad = (-1)^{u-1}\left(\tfrac{m}{n}\right)^{p-u}\left[\binom{p}{u-1}\left(\tfrac{m}{n}\right)(p-u+1)n - \binom{p}{u}um\right] \\ &\qquad = 0,\end{aligned}$$

where the last equality can be verified easily. We deduce that the right-hand side of Equation (5) is zero.

Next, let $r \in \{-1, 0, \ldots, 2p-1\}$, and let $G_r$ denote the coefficient of $b_r$ in the left-hand side of Equation (5).

Because $f_{-1}(T_{p-w,w}) = 1$ for all $w \in \{0, 1, \ldots, p\}$, we see that

$$G_{-1} = \sum_{w=0}^{p}(-1)^w \binom{p}{w}\left(\tfrac{m}{n}\right)^{p-w} = \left(\tfrac{m}{n} - 1\right)^p.$$

Now suppose that $r \neq -1$. Then, using Equation (4), we see that

$$\begin{aligned}
G_r &= \sum_{w=0}^{p} (-1)^w \binom{p}{w} \left(\frac{m}{n}\right)^{p-w} f_r(T_{p-w,w}) \\
&= \sum_{w=0}^{p} (-1)^w \binom{p}{w} \left(\frac{m}{n}\right)^{p-w} \sum_{i=-1}^{r} \sum_{j=0}^{p-w} \sum_{k=0}^{w} \binom{p-w}{j} \binom{j}{2j+i-r} \binom{w}{k} \binom{k}{2k-i-1} n^j m^k. \\
&= \sum_{w=0}^{p} \sum_{i=-1}^{r} \sum_{j=0}^{p-w} \sum_{k=0}^{w} (-1)^w \binom{p}{w} \binom{p-w}{j} \binom{j}{2j+i-r} \binom{w}{k} \binom{k}{2k-i-1} \frac{m^{p-w+k}}{n^{p-w-j}}. \qquad (6)
\end{aligned}$$

Observe that $G_r$ is a Laurent polynomial in $m$ and $n$. Let $w \in \{0, 1, \ldots, p\}$, and $j \in \{0, 1, \ldots, p-w\}$ and $k \in \{0, 1, \ldots, w\}$. We will use the substitution $a = p - w + k$ and $b = p - w - j$. It is seen that $0 \leq a \leq p$, and $0 \leq b \leq p$, and $a - (p - w) = k \geq 0$ and $(p - w) - b = j \geq 0$. It is evident that $a \geq b$. In fact, there is no term in $G_r$ that has $\frac{m^a}{n^b}$ with $a = b$. Suppose the contrary. Because $k, j \geq 0$, the only values of $k$ and $j$ that would yield $a = b$ are $k = 0 = j$. If that were the case, then $\binom{j}{2j+i-r}\binom{k}{2k-i-1} = \binom{0}{i-r}\binom{0}{-i-1}$. Note that $i \in \{-1, 0, \ldots, r\}$. If $i < r$, then $\binom{0}{i-r} = 0$, and if $i > -1$, then $\binom{0}{-i-1} = 0$. Hence, the coefficients of $\frac{m^a}{n^b}$ with $a = b$ are all zero, and we may therefore restrict our attention to the case where $a > b$.

Let $D_{a,b}$ denote the coefficient of $\frac{m^a}{n^b}$ in Equation (6). For each possible value of $w$, there is one choice of each of $k$ and $j$ that yield the desired powers of $m$ and $n$. Specifically, it is seen that

$$D_{a,b} = \sum_{w=0}^{p} (-1)^w \binom{p}{w} \binom{p-w}{(p-w)-b} \binom{w}{a-(p-w)} \sum_{i=-1}^{r} \binom{(p-w)-b}{2(p-w)-2b+i-r} \binom{a-(p-w)}{2a-2(p-w)-i-1}. \qquad (7)$$

Recalling that $a - (p - w) \geq 0$ and $(p - w) - b \geq 0$, we observe that if $i \in \mathbb{Z}$, then $i < -1$ implies $\binom{a-(p-w)}{2a-2(p-w)-i-1} = 0$, and $i > r$ implies $\binom{(p-w)-b}{2(p-w)-2b+i-r} = 0$. Hence, in Equation (7), we can replace the upper and lower limits of the inner sum, which are $i = -1$ and $i = r$, with any lower limit less than $-1$ and any upper limit greater than $r$. By doing so if needed, we can apply Vandermonde's Convolution Formula to deduce, as the reader can verify, that

$$\sum_{i=-1}^{r} \binom{(p-w)-b}{2(p-w)-2b+i-r} \binom{a-(p-w)}{2a-2(p-w)-i-1} = \binom{a-b}{2(a-b)-r-1}. \qquad (8)$$

Next, using the definition of binomial coefficients, it is easy to verify that

$$\binom{p}{w}\binom{p-w}{(p-w)-b}\binom{w}{a-(p-w)} = \binom{p}{a-b}\binom{p-(a-b)}{b}\binom{a-b}{a-(p-w)}. \tag{9}$$

Combining Equations (7), (9) and (8), using the substitution $z = a - (p-w)$, and doing some rearranging yields

$$D_{a,b} = \binom{p}{a-b}\binom{p-(a-b)}{b}\binom{a-b}{2(a-b)-r-1}\sum_{z=a-p}^{a}(-1)^{p-(a-z)}\binom{a-b}{z}.$$

Observe that $a-p \leq 0 < a-b \leq a$. Hence, the only values of $z$ for which $\binom{a-b}{z}$ is non-zero are $z \in \{0,\ldots,a-b\}$. Then

$$\sum_{z=a-p}^{a}(-1)^{p-(a-z)}\binom{a-b}{z} = (-1)^{p-a}\sum_{z=0}^{a-b}(-1)^{z}\binom{a-b}{z} = (-1)^{p-a}(1-1)^{a-b} = 0.$$

Therefore $D_{a,b} = 0$.

Putting the above together, we see that Equation (5) reduces to the very simple equation $\left(\frac{m}{n}-1\right)^{p} b_{-1} = 0$. Given that $m \neq n$ and $b_{-1} \neq 0$, we have reached a contradiction, from which we deduce that $\Lambda$ is not locally determined. □

Part (1) of Theorem 2.1 would occur, for example, when the set $\mathcal{T}$ is the set of all $m$-spheres for some even $m \in \mathbb{N}$.

We note that whereas in Part (3) of the theorem it is hypothesized that $\mathcal{T}$ contains all flag $d$-spheres for some odd integer $d$ such that $d \geq 3$, it is seen in the proof of the theorem that not all flag $d$-spheres are needed, but rather, by using $n = 4$ and $m = 5$, it would suffice to include only those flag $d$-spheres with up to $\frac{5(d+1)}{2}$ vertices. The proof of the theorem was given with arbitrary $n$ and $m$, rather than only $n = 4$ and $m = 5$, because it is easier to see what is going on by treating the more general case.

The three cases in Theorem 2.1 do not exhaust all possibilities. However, Part (3) suffices to treat both the reduced Euler characteristic, which is $\tilde{\chi}(K) = \sum_{i=-1}^{\dim K}(-1)^i f_i(K)$, and the Charney-Davis quantity, as we now state.

**Corollary 2.2.** *Let $\mathcal{T}$ be a set of finite simplicial complexes that contains all flag $d$-spheres for some odd integer $d$ such that $d \geq 3$. Then both the reduced Euler characteristic $\tilde{\chi}$ and the Charney-Davis function $\lambda$ on $\mathcal{T}$ are not combinatorially locally determined.*

## 3 Geometrically Locally Determined Functions

For the geometric approach of [4], we need the following definitions (which, in contrast to the original, are given here for arbitrary dimensions). Recall that in this approach, we consider geometric simplicial complexes that are embedded in Euclidean space, where each simplex is the convex hull of its vertices, and hence the embedding is determined by the locations of its vertices. We view different embeddings of the same abstract simplicial complex as different simplicial complexes.

**Definition.** Let $\mathcal{T}$ be a set of simplicial complexes. A **real-valued vertex-supported function** on $\mathcal{T}$ is a function $\phi$ that assigns to every $K \in \mathcal{T}$, and every $v \in K^{(0)}$, a real number $\phi(v, K)$. △

For the following definition, suppose that $K$ and $\{K_n\}_{n=1}^{\infty}$ are combinatorially equivalent simplicial complexes, and all are embedded in the same Euclidean space. We can think of these simplicial complexes as embeddings of the same abstract simplicial complex. We write $\lim_{n\to\infty} K_n = K$ to denote pointwise convergence of these embeddings; it suffices to verify convergence at the vertices of the abstract simplicial complex.

**Definition.** Let $\mathcal{T}$ be a set of simplicial complexes, and let $\phi$ be a real-valued vertex-supported function on $\mathcal{T}$.

1. The function $\phi$ is **invariant under subdivision** if the following condition holds. If $K, J \in \mathcal{T}$, where $J$ is a subdivision of $K$, and if $v \in K^{(0)}$, then $\phi(v, K) = \phi(v, J)$.

2. The function $\phi$ is **invariant under simplicial isometries of stars** if the following condition holds. If $K, L \in \mathcal{T}$, if $v \in K^{(0)}$ and $w \in L^{(0)}$, and if there is a simplicial isometry $|\operatorname{star}(v, K)| \to |\operatorname{star}(w, L)|$ that takes $v$ to $w$, then $\phi(v, K) = \phi(w, L)$.

3. The function $\phi$ is **continuous** if the following condition holds. Let $K$ and $\{K_n\}_{n=1}^{\infty}$ be combinatorially equivalent simplicial complexes in $\mathcal{T}$, all embedded in the same Euclidean space. Suppose $\lim_{n\to\infty} K_n = K$. If $v \in K^{(0)}$, and if the corresponding vertex of $K_n$ is labeled $v_n$, then $\lim_{n\to\infty} \phi(v_n, K_n) = \phi(v, K)$. △

**Definition.** Let $\mathcal{T}$ be a set of simplicial complexes, and let $\Lambda$ be a real-valued function on $\mathcal{T}$. The function $\Lambda$ is **geometrically locally determined** by a real-valued vertex-supported function $\phi$ on $\mathcal{T}$ if $\phi$ is invariant under simplicial isometries of stars, is invariant under subdivision and is continuous, and if

$$\Lambda(K) = \sum_{v \in K^{(0)}} \phi(v, K)$$

for every $K \in \mathcal{T}$. $\triangle$

For the proof of Part (1) of Theorem 3.1 below, which is a simple variation of an argument in [3], we adopt the convention that all angles are normalized so that the volume of the unit $(n-1)$-sphere in $(n-1)$-measure is 1 in all dimensions. For any $n$-simplex $\sigma^n$ in Euclidean space, and any $i$-face $\eta^i$ of $\sigma^n$, let $\alpha(\eta^i, \sigma^n)$ denote the solid angle in $\sigma^n$ along $\eta^i$, where by normalization such an angle is a number in $[0, 1]$.

We make use here of a lemma, found in many places and stated as Lemma 3.1 in [3], which generalizes the fact that the angles of a planar triangle add up to $\pi$ (or $1/2$ when normalized); the lemma reduces to that result when $n = 2$. A simple proof of this lemma appears on p. 24 of [12]; for historical remarks, see p. 312 of [9].

Our result in the geometric setting is the following.

**Theorem 3.1.** *Let $\mathcal{T}$ be a set of finite simplicial complexes, and let $\Lambda$ be a real-valued function on $\mathcal{T}$. Suppose that $\Lambda$ has the form $\Lambda(K) = \sum_{i=-1}^{\dim K} b_i f_i(K)$ for all $K \in \mathcal{T}$, for some $b_{-1}, b_0, b_1, \ldots \in \mathbb{R}$.*

1. *If $b_{-1} = 0$, and if $\mathcal{T}$ is a set of $n$-dimensional pseudomanifolds for some integer $n$ such that $n \geq 2$, then $\Lambda$ is geometrically locally determined.*

2. *If $b_{-1} \neq 0$, and if $\mathcal{T}$ contains all flag $d$-spheres for some odd integer $d$ such that $d \geq 3$, then $\Lambda$ is not geometrically locally determined.*

*Proof.* For Part (1), let $K \in \mathcal{T}$. Because $K$ is an $n$-dimensional pseudomanifold, we have $(n+1)f_n(K) = 2f_{n-1}(K)$.

We use the notation $\eta^i$ to denote an $i$-simplex of $K$. Let

$$P = \frac{2(-1)^n}{n-1}\left[\frac{(n+1)}{2} b_{n-1} + b_n\right].$$

For each $v \in K^{(0)}$, let

$$\phi(v, K) = \sum_{i=0}^{n-2} \frac{1}{i+1} \sum_{\eta^i \ni v} \left[b_i + (-1)^i P \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n)\right].$$

Because $\phi$ is determined by angle sums of the form $\sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n)$, it is invariant under simplicial isometries of stars, is invariant under subdivision and is continuous.

We compute

$$\begin{aligned}
\sum_{v \in K^{(0)}} \phi(v, K) &= \sum_{v \in K^{(0)}} \sum_{i=0}^{n-2} \frac{1}{i+1} \sum_{\eta^i \ni v} \left[ b_i + (-1)^i P \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n) \right] \\
&= \sum_{i=0}^{n-2} \frac{1}{i+1} \sum_{\eta^i \in K} \sum_{v \in \eta^i} \left[ b_i + (-1)^i P \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n) \right] \\
&= \sum_{i=0}^{n-2} \sum_{\eta^i \in K} \left[ b_i + (-1)^i P \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n) \right]
\end{aligned}$$

because $\eta^i$ has $i+1$ vertices

$$\begin{aligned}
&= \sum_{i=0}^{n-2} b_i f_i(K) + P \sum_{i=0}^{n-2} \sum_{\eta^i \in K} \sum_{\sigma^n \succ \eta^i} (-1)^i \alpha(\eta^i, \sigma^n) \\
&= \sum_{i=0}^{n-2} b_i f_i(K) + P \sum_{\sigma^n \in K} \sum_{\substack{\eta^i \prec \sigma^n \\ 0 \le i \le n-2}} (-1)^i \alpha(\eta^i, \sigma^n) \\
&= \sum_{i=0}^{n-2} b_i f_i(K) + P \sum_{\sigma^n \in K} \frac{(-1)^n (n-1)}{2}
\end{aligned}$$

by Lemma 3.1 of [3]

$$\begin{aligned}
&= \sum_{i=0}^{n-2} b_i f_i(K) + P \frac{(-1)^n (n-1)}{2} f_n(K) \\
&= \sum_{i=0}^{n-2} b_i f_i(K) + \frac{(n+1)}{2} b_{n-1} f_n(K) + b_n f_n(K) \\
&= \sum_{i=0}^{n} b_i f_i(K).
\end{aligned}$$

Hence $\Lambda$ is geometrically locally determined by $\phi$.

For Part (2), we simply need to modify the proof of Part (3) of Theorem 2.1 very slightly, as follows. First, embed each copy of $C_n$ or $C_m$ in

$\mathbb{R}^2$ by having the vertices be on the unit circle in $\mathbb{R}^2$, equally spaced, and then construct $T_{p-u,u}$ in $(\mathbb{R}^2)^p = \mathbb{R}^{2p}$. It is then seen that all the vertices in $T_{p-u,u}$ that are in copies of $C_n$ have isometric stars, and similarly for all the vertices in $T_{p-u,u}$ that are in copies of $C_m$. Hence, if $\Lambda$ were geometrically locally determined by a real-valued vertex-supported function $\phi$, and if $\phi$ is assumed to be invariant under simplicial isometries of stars (it does not even have to satisfy the other two conditions in the definition of geometrically locally determined), then the same example used in the proof of Part (3) of Theorem 2.1, but with $h(\text{link}\,(v, T_{s,t}))$ replaced by $\phi(v, T_{s,t})$, will yield the same contradiction. □

The following corollary is immediate.

**Corollary 3.2.** *Let $\mathcal{T}$ be a set of finite simplicial complexes that contains all flag $d$-spheres for some odd integer $d$ such that $d \geq 3$. Then both the reduced Euler characteristic $\tilde{\chi}$ and the Charney-Davis function $\lambda$ on $\mathcal{T}$ are not geometrically locally determined.*

The proof of Part (1) of Theorem 3.1 works for $n$-dimensional pseudomanifolds but not for all simplicial complexes. We note, however, that it is possible to modify the definition of $\phi(v, K)$ in the proof in such a way that it works for all pure finite $n$-dimensional simplicial complexes, though at the cost that instead of obtaining expressions of the form $\sum_{i=0}^{n} b_i f_i(K)$, each number $f_i(K)$ would be replaced by a variant of it that is weighted by the extent to which $K$ is not a pseudomanifold, using the methodology of [3]. We omit the details.

Finally, we note that in Section 2 of [4], it was mistakenly claimed that the function $\Lambda$ on the set of all 2-dimensional simplicial complexes defined by $\Lambda(K) = f_2(K)$ for all $K$ is not geometrically locally determined. It is seen by Part (1) of Theorem 3.1 that this function $\Lambda$ is geometrically locally determined on the set of all finite 2-dimensional pseudomanifolds, and, using the ideas of the above remark, it can be verified that $\Lambda$ is geometrically locally determined on the set of all 2-dimensional simplicial complexes.

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